CYBERNETICS AND CONTROL THEORY

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Abstract

Full Text

CYBERNETICS AND CONTROL THEORY

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ON VALVE CIRCUITS

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Let us consider the realization of Boolean matrices by valve circuits*. We shall call the depth of a valve circuit the greatest of the lengths of chains connecting the input and output poles of the circuit. Denote by $B_m(p,q)$ Shannon's function for realizations of (p,q)-matrices** by circuits of depth not exceeding m, and by $B(p,q)$ Shannon' s function for realizations by circuits of arbitrary depth. In ([1]) an asymptotic estimate was obtained for

$$B(p_n, q_n)$$

under the condition

$$v_n = \frac{\lg_2 q_n}{\lg_2 p_n} \to 0$$

(and some others). At the same time it turns out that $B(p_n,q_n) \sim B_2(p_n,q_n)$. Below we succeed in obtaining the estimate $B(p_n,q_n) \sim B_3(p_n,q_n)$ under the condition that $\lim v_n$ exists and belongs to some nowhere dense set of points of the interval $[0,1]$.

Estimates of the complexity of valve circuits have numerous applications; for example, the realization of linear transformations is almost uniquely described by valve circuits.

1°. Denote by $\chi(\mathfrak{A})$ the matrix realized by the circuit $\mathfrak{A}$. Let $\mathfrak{A}'$ be a (p,r)-circuit and $\mathfrak{A}''$ an (r,q)-circuit. Denote by $\mathfrak{A}' \times \mathfrak{A}''$ the (p,q)-circuit obtained as a result of identifying the l-th output pole of the circuit $\mathfrak{A}'$ with the l-th input pole of the circuit $\mathfrak{A}''$ $(l = 1, \dots, r)$.

Lemma.

$$\chi(\mathfrak{A}' \times \mathfrak{A}'') = \chi(\mathfrak{A}') \times \chi(\mathfrak{A}'').$$

(The symbol $\times$ also denotes multiplication of matrices.)

Denote by $\|A\|$ the number of ones in the matrix A. Denote by $\mathfrak{B}(p_n, q_n, \alpha_n)$ the class of all Boolean (p_n, q_n)-matrices A such that $\|A\| = \alpha_n p_n q_n$ (the conditions $0 \leq \alpha_n \leq 1$ and that $\alpha_n p_n q_n$ is a natural number will be understood henceforth without explicit mention).

Theorem 1. Suppose the following conditions are satisfied:

a) $q_n \leq p_n$;

b) $\alpha_n q_n \to \infty$;

c) $\lg_2 p_n / \lg_2 \dfrac{1}{\alpha_n} \to \rho$, where ρ is an integer greater than zero;

d) $p_n \alpha_n^\rho \to \infty$.

Then for $\mathfrak{B}(p_n, q_n, \alpha_n)$

$$B_2(p_n, q_n) \sim \frac{\alpha_n p_n q_n}{\rho}.$$

Proof. The lower bound is obtained as in ([1]).

Upper bound. Let $D_n \in \mathfrak{B}(p_n, q_n, \gamma_n)$, $\beta_n \leq \gamma_n \leq \alpha_n$, where β_n is an arbitrary parameter satisfying the conditions

$$\frac{\beta_n}{\alpha_n} \to 0, \qquad \beta_n q_n \to \infty, \qquad p_n \beta_n^\rho \to \infty.$$

* In the present paper the definition of a valve (p, q)-circuit introduced in ([1]) is used.

** By a (p, q)-matrix we mean a matrix having p rows and q columns.

If n is sufficiently large, then in the matrix D_n there exists a (t_n, ρ)-submatrix consisting entirely of ones, with $t_n \to \infty$. Indeed, the number of all $(1, \rho)$-submatrices of the matrix D_n consisting entirely of ones is not less than $p_n C^\rho_{[\gamma_n q_n]}$ (because the smallest number of such submatrices occurs in the case of a uniform distribution of the ones among the rows of the matrix D_n). Let us call the type of a $(1, \rho)$-submatrix the system of numbers of the columns of the matrix D_n in which this submatrix is located. The t_n different $(1, \rho)$-submatrices of one type form a (t_n, ρ)-submatrix. Since the number of types is equal to $C^\rho_{q_n}$, there exists at least one type to which belong at least

$$t_n = \left] \frac{p_n C^\rho_{[\gamma_n q_n]}}{C^\rho_{q_n}} \right[$$

$(1, \rho)$-submatrices consisting entirely of ones. $t_n \sim p_n \gamma_n^\rho$, since $\gamma_n q_n \to \infty$.

We describe the process of constructing the matrices $D_n^{(k)}$, $k = 1, \dots, N_n(\beta_n)+1$. Denote by $\gamma_n^{(k)}$ the number $\|D_n^{(k)}\|/p_n q_n$. Put $D_n^{(1)} = A_n$. If $\beta_n \leqslant \gamma_n^{(k)}$, then, by the preceding, we select in the matrix $D_n^{(k)}$ a $(t_n^{(k)}, \rho)$-submatrix consisting entirely of ones; denote it by $a_n^{(k)}$. Denote by $A_n^{(k)}$ the matrix obtained from $D_n^{(k)}$ by replacing all elements by zeros except the elements of the submatrix $a_n^{(k)}$, and by $D_n^{(k+1)}$ the matrix obtained from $D_n^{(k)}$ by replacing by zeros all elements of the submatrix $a_n^{(k)}$; we have $D_n^{(k)} = A_n^{(k)} \vee D_n^{(k+1)}$. If $\gamma_n^{(k)} < \beta_n$, then we put $k = N_n(\beta_n) + 1$. Thus, the process consists in successively selecting submatrices filled with ones; the matrices formed become increasingly "sparse," and the process terminates as soon as a matrix is formed containing fewer than $\beta_n p_n q_n$ ones.

Denote by H_n the matrix $D_n^{(N_n(\beta_n)+1)}$. We have $A_n = A_n^{(1)} \vee \dots \vee A_n^{(N_n(\beta_n))} \vee H_n$, $\min_{k=1,\dots,N_n(\beta_n)} t_n^{(k)} \gtrsim p_n \beta_n^\rho$, $\|H_n\| < \beta_n p_n q_n$. Represent each matrix $A_n^{(k)}$ in the form $A_n^{(k)} = F_n^{(k)} \times G_n^{(k)}$, where $F_n^{(k)}$ is a $(p_n, 1)$-matrix, $G_n^{(k)}$ is a $(1, q_n)$-matrix, $\|F_n^{(k)}\| = t_n^{(k)}$, $\|G_n^{(k)}\| = \rho$, and let

$$F_n = (F_n^{(1)}, \dots, F_n^{(N_n(\beta_n))}), \qquad G_n = \begin{pmatrix} G_n^{(1)} \\ \vdots \\ G_n^{(N_n(\beta_n))} \end{pmatrix}.$$

Then

$$A_n = F_n \times G_n \vee H_n, \tag{1}$$

$$\|F_n\| \leqslant \frac{\alpha_n p_n q_n}{\rho}, \qquad \|G_n\| \leqslant \frac{\alpha_n q_n}{\beta_n^\rho}, \qquad \|H_n\| < \beta_n p_n q_n. \tag{2}$$

Realizing each matrix F_n, G_n, H_n by a circuit of depth 1, we obtain, by the lemma, a circuit for A_n. By (2), $B_2(p_n, q_n) \lesssim \alpha_n p_n q_n/\rho$. The theorem is proved.

Remark. If conditions a), b), d) are satisfied and in condition c) ρ is not an integer, then

$$B_2(p_n, q_n) \lesssim \frac{\alpha_n p_n q_n}{[\rho]}.$$

It can be shown that in this case the trivial power lower bound is ineffective, i.e., it can be improved.

2°. **Theorem 2.** Suppose the following conditions are satisfied:

a) $q_n \leqslant p_n$;

b) $q_n \to \infty$;

c)

$$\lim_{n\to\infty} \frac{\lg_2 q_n}{\lg_2 p_n} = \frac{\mu}{\mu(\rho-1)+\rho},$$

where μ, ρ are positive integers.*

Then

$$B(p_n, q_n) \sim B_3(p_n, q_n) \sim \frac{p_n q_n}{\lg_2(p_n q_n)}.$$

Proof. The lower estimate is the cardinality estimate.

The upper estimate. Let A_n be a (p_n, q_n)-matrix. Introduce the parameter ϑ_n and divide the matrix A_n into

$$T_n = \left]\frac{q_n}{\vartheta_n}\right[$$

nonoverlapping $(p_n, \vartheta_{n,k})$-submatrices

$$A_n^{(k)}, \qquad A_n = (A_n^{(1)}, \dots, A_n^{(T_n)}),$$

so that $\vartheta_{n,k} = \vartheta_n$ for $k = 1, \dots, T_n - 1$ and $\vartheta_{n,T_n} \leqslant \vartheta_n$. Form a $(2^{\vartheta_{n,k}}, \vartheta_{n,k})$-matrix $\Sigma_{n,k}$, whose rows are all possible distinct vectors, taken in arbitrary order. Obviously, there exist $(p_n, 2^{\vartheta_{n,k}})$-matrices $B^{(k)}(A_n)$, having exactly one one in each row, such that

$$A_n^{(k)} = B^{(k)}(A_n) \times \Sigma_{n,k}, \qquad k = 1, \dots, T_n.$$

Introduce the matrices

$$B(A_n) = (B^{(1)}(A_n), \dots, B^{(T_n)}(A_n)),$$

$$\Sigma_n = \begin{pmatrix} \Sigma_{n,1} & & 0 \\ & \ddots & \\ 0 & & \Sigma_{n,T_n} \end{pmatrix}.$$

Then

$$A_n = B(A_n) \times \Sigma_n.$$

Denote by q'_n the number

$$\sum_{k=1}^{T_n} 2^{\vartheta_{n,k}}.$$

Introduce the parameter λ_n and divide the matrix $B(A_n)$ into

$$U_n = \left]\frac{p_n}{\lambda_n}\right[$$

nonoverlapping $(\lambda_{n,i}, q'_n)$-submatrices

$$B_i(A_n),$$

$$B(A_n) = \begin{pmatrix} B_1(A_n) \\ \vdots \\ B_{U_n}(A_n) \end{pmatrix}$$

so that $\lambda_{n,i} = \lambda_n$ for $i = 1, \dots, U_n - 1$ and $\lambda_{n,U_n} \leqslant \lambda_n$. Form a $(\lambda_{n,i}, C^{\mu+1}_{\lambda_{n,i}})$-matrix $\mathfrak{S}^{(n,i)}$, whose columns are all possible distinct vectors containing exactly $\mu + 1$ ones, taken in arbitrary order (if $\lambda_{n,i} < \mu + 1$, then we put $C^{\mu+1}_{\lambda_{n,i}} = 0$). Obviously, there exist $(C^{\mu+1}_{\lambda_{n,i}}, q'_n)$-matrices $C_i(A_n)$ $(i = 1, \dots, U_n)$ such that

$$B_i(A_n) = \mathfrak{S}^{(n,i)} \times C_i(A_n) \vee B^*_i(A_n),$$

where

$$\|B^*_i(A_n)\| \leqslant (\mu + 1)q'_n, \qquad \frac{\lambda_n T_n}{\mu + 1} - q'_n \leqslant \|C_i(A_n)\| \leqslant \frac{\lambda_n T_n}{\mu + 1}$$

(the product of an $(a, 0)$-matrix by a $(0, b)$-matrix is defined as an (a, b)-matrix consisting entirely of zeros). Introduce the matrices

$$\mathfrak{S}^{(n)} = \begin{pmatrix} \mathfrak{S}^{(n,1)} & & 0 \\ & \ddots & \\ 0 & & \mathfrak{S}^{(n,U_n)} \end{pmatrix}, \qquad C(A_n) = \begin{pmatrix} C_1(A_n) \\ \vdots \\ C_{U_n}(A_n) \end{pmatrix}, \qquad B^*(A_n) = \begin{pmatrix} B^*_1(A_n) \\ \vdots \\ B^*_{U_n}(A_n) \end{pmatrix}.$$

* Condition c) includes the important case for applications when $p_n \succ q_n$.

Then

$$B(A_n) = \mathfrak{S}^{(n)} \times C(A_n) \vee B^*(A_n). \tag{4}$$

Denote by p'_n the number $\sum_{i=1}^{U_n} C^{\mu+1}_{\lambda_n,i}$; $C(A_n)$ is a (p'_n, q'_n)-matrix; denote by α_n the number $\|C(A_n)\|/p'_n q'_n$, and by v_n the number $\lg_2 q_n/\lg_2 p_n$.

Introduce, for the parameters ϑ_n, λ_n, the conditions: 1) $\lambda_n/2^{\vartheta_n} \to \infty$; 2) $q_n/\vartheta_n\lambda_n^\mu \to \infty$. Then

$$T_n \sim q_n/\vartheta_n, \quad q'_n \sim q_n 2^{\vartheta_n}/\vartheta_n, \quad U_n \sim p_n/\lambda_n, \quad p'_n \sim p_n\lambda_n^\mu/(\mu{+}1)!, \quad \|C(A_n)\| \sim p_n q_n/(\mu{+}1)\vartheta_n,$$

$$\alpha_n \sim \mu!/2^{\vartheta_n}\lambda_n^\mu, \quad \alpha_n q'_n \to \infty.$$

Introduce, further, the conditions: 3) $\lg_2 p'_n \big/ \lg_2 \dfrac{1}{\alpha_n} \to \rho$; 4) $p'_n \alpha_n^\rho \to \infty$.

Introduce the parameter β_n and the conditions: 5) $\beta_n 2^{\vartheta_n} \lambda_n^\mu \to 0$; 6) $\beta_n q_n 2^{\vartheta_n}/\vartheta_n \to \infty$; 7) $p_n \lambda_n^\mu \beta_n^\rho \to \infty$.

Apply to $C(A_n)$ the construction of Theorem 1; by virtue of (1), (3), (4),

$$C(A_n) = F_n \times G_n \vee H_n$$

and

$$A_n = \mathfrak{S}^{(n)} \times F_n \times (G_n \times \Sigma_n) \vee (\mathfrak{S}^{(n)} \times H_n \vee B^*(A_n)) \times \Sigma_n.$$

We obtain the circuit for A_n from circuits of depth 1 for $F_n, \mathfrak{S}^{(n)}, G_n \times \Sigma_n, H_n, B^*(A_n), \Sigma_n$ (lemma). We have

$$\|\mathfrak{S}^{(n)}\| \leq p_n \lambda_n^\mu, \quad \|\Sigma_n\| \leq q_n 2^{\vartheta_n}, \quad \|B^*(A_n)\| \leq \frac{p_n q_n 2^{\vartheta_n}}{\vartheta_n \lambda_n},$$

and, by virtue of (2),

$$\|F_n\| \leq p_n q_n (\mu+1)\rho^{\vartheta_n}, \quad \|G_n \times \Sigma_n\| \leq \vartheta_n \|G_n\| \leq \frac{q_n}{\lambda_n^\mu p_n^{\beta_n^\rho}},$$

$$\|H_n\| \leq \frac{\beta_n p_n q_n 2^{\vartheta_n} \lambda_n^\mu}{\vartheta_n}.$$

In order that the relations

$$B_3(p_n, q_n) \sim \|F_n\| \leq p_n q_n / \lg_2(p_n q_n)$$

hold, it is sufficient that the following additional conditions hold:

8) $(\mu+1)\rho^{\vartheta_n} \geq (1+v_n)\lg_2 p_n$; 9) $\lambda_n^\mu \lg_2 p_n / q_n \to 0$; 10) $2^{\vartheta_n} \lg_2 p_n / p_n \to 0$; 11) $2^{\vartheta_n} \lg_2 p_n / \vartheta_n \lambda_n \to 0$; 12) $\lg_2 p_n / p_n \lambda_n^\mu \beta_n^\rho \to 0$; 13) $\beta_n 2^{\vartheta_n} \lambda_n^\mu \lg_2 p_n / \vartheta_n \to 0$.

By the conditions of the theorem,

$$v_n = \frac{\mu}{\mu(\rho-1)+\rho} + \varphi_n,$$

where $\varphi_n \to 0$. Put, for sufficiently large n,

$$\lambda_n = \left[p_n^{\frac{1}{\mu(\rho-1)+\rho}+\delta_n}\right], \quad \vartheta_n = \left[\left(\frac{1}{\mu(\rho-1)+\rho} + \xi_n\right)\lg_2 p_n\right];$$

$$\frac{1}{\beta_n} = p_n^{\frac{\mu+1}{\mu(\rho-1)+\rho}+\eta_n};$$

$$\delta_n = \frac{\varphi_n - 3\lg_2 \lg_2 p_n / \lg_2 p_n}{\mu}; \quad \xi_n = -3\frac{\lg_2 \lg_2 p_n}{\lg_2 p_n} + \min\left[\delta_n, \mu\left(\frac{1}{\rho}-1\right)\delta_n\right];$$

$$\eta_n = \mu\delta_n + \xi_n + \frac{\lg_2 \lg_2 p_n}{\lg_2 p_n}.$$

Then conditions 1)—13) are fulfilled. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.