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Abstract

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THE FIRST BOUNDARY VALUE PROBLEM FOR NONLINEAR PARABOLIC EQUATIONS

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1. In this paper the general nonlinear parabolic equation of second order is considered

$$u_t - F(x, t, u, u_{x_k}, u_{x_k x_l}) = 0 \qquad (k, l = 1, 2, \dots, n). \tag{1}$$

In the cylindrical domain $Q_{t_0 T} = \{\overline{G} \times [t_0 T]\}$, $x \in G,\ t \in [t_0 T]$, where G is a bounded domain of n-dimensional space with boundary S, the first boundary value problem is posed:

$$u\Big|_{t=t_0} = u_0(x); \qquad u\Big|_{\Gamma} = \varphi(x, t), \quad \text{where } \Gamma = \{S \times [t_0 T]\}. \tag{2}$$

The function $F(x, t, u, p_k, p_{kl})$ is assumed to be defined for $(x, t) \in Q_{t_0 T}$ and for arbitrary values of the remaining arguments.

The purpose of the present paper is to establish a theorem on the local solvability of problem (1)–(2) under only certain smoothness conditions imposed on the function $F(x, t, u, p_k, p_{kl})$, the boundary S, and the boundary functions $u_0(x)$ and $\varphi(x, t)$. The exact formulation of these conditions will be given below. It should be noted that no conditions on the growth of the function $F(x, t, u, p_k, p_{kl})$ with respect to the arguments u, p_k, p_{kl} $(k, l = 1, \dots, n)$ are imposed.

An analogous theorem for quasilinear equations was established by P. E. Sobolevskii (^{1}) with the aid of the theory of fractional powers of operators and the Schauder principle.

The solution of problem (1)–(2) is constructed in the class $C^{2+\alpha}$ by the method of successive approximations, analogous to Picard' s method for ordinary equations. This construction essentially relies on Friedman' s results (^{3}) on an a priori estimate in the closed domain of the solution of the first boundary value problem and on the solvability of such a problem for linear parabolic equations. The uniqueness theorem for problem (1)–(2) was established in (^{2}).

Let us make several remarks concerning notation. $|u|_{k+\alpha}^{st}$ denotes the norm in the space $C^{k+\alpha}(Q_{st})$, where $Q_{st} = \{\overline{G} \times [s,t]\}$; $k = 0,1,2;\ 0 \le \alpha < 1$.

The spaces $C^{2+\alpha}(G)$ and $C^{2+\alpha}(Q_{tt})$ are distinguished. The latter denotes the space of limit values of functions depending on t, and the norms in these spaces are related by the relation

$$|u|_{2+\alpha}^{tt} = |u|_{2+\alpha}^{G} + |u_t|_{\alpha}^{G}.$$

The class of functions $u(x,t)$ from $C^{2+\alpha}(Q_{t_0\tau})$ satisfying conditions (2) for $t_0 \le t \le \tau$, if, of course, conditions (2) allow this, is denoted by $C^{t_0\tau}$.

2. We formulate the main result of the paper.

Theorem. *Let the following conditions be satisfied:*

1) In any domain

$$R_a = \left\{(x,t) \in Q_{t_0T},\ |u| + \sum |p_i| + \sum |p_{ij}| \le a\right\}$$

the function $F(x,t,u,p_k,p_{kl})$ has bounded third derivatives with respect to u, p_k, p_{kl} $(k,l = 1,2,\dots,n)$, and F_t has bounded second derivatives with respect to the same arguments. Moreover, F, F_t, and all the derivatives listed satisfy a Hölder condition in (x,t) with exponent α, $0 < \alpha < 1$.

2) The parabolicity condition for equation (1) in the domains R_a is fulfilled in the form

$$\sum_{ij} F_{p_{ij}} \xi_i \xi_j \ge \nu(a) \sum_i \xi_i^2,$$

where $\nu(a)$ is a positive nonincreasing function.

3) $S \in A^{2+\alpha}$ (see (3)) and there exist functions $\psi_1 \in C^{2+\alpha}(Q_{t_0T})$ and $\psi_2 \in C^{2+\alpha}(Q_{t_0T})$ such that

$$\psi_1(x,t_0) = u_0(x); \qquad \psi_1(x,t)\big|_{\Gamma} = \varphi(x,t);$$

$$\psi_2(x,t_0) = w_0(x) = F(x,t_0,u_0(x),u_{0x_i},u_{0x_ix_j}); \qquad \psi_2(x,t)\big|_{\Gamma} = \varphi_t(x,t).$$

Moreover, the following compatibility condition is satisfied:

$$\varphi_{tt}(x,t_0) = \sum_{ij} F_{p_{ij}}(x,t_0,u_0,u_{0x_k},u_{0x_kx^l})\, w_{0x_ix_j} +$$

$$+\sum_i F_{p_i}(\cdot)w_{0x_i} + F_u(\cdot)w_0 + F_t(\cdot)$$

for $x \in S$; $(\cdot)$ denotes the same arguments as for $F_{p_{ij}}$.

Under these conditions there exists a $t' > t_0$ $(t' \le T)$ such that problem (1)–(2) has in the domain $Q_{t_0 t'}$ a unique solution $u(x,t)$ such that $u \in C^{2+\alpha}(Q_{t_0 t'})$, $u_t \in C^{2+\alpha}(Q_{t_0 t'})$.

To prove the theorem just formulated, we consider the auxiliary boundary-value problem for the linear equation:

$$w_t - \sum_{ij} F_{p_{ij}}(x,t,u,u_{x_k},u_{x_{kx}l})w_{x_i x_j} - \sum_i F_{p_i}(\cdot)w_{x_i} - F_u(\cdot)w - F_t(\cdot); \qquad (3)$$

$$w\big|_{t=t_0} = w_0(x); \qquad w\big|_\Gamma = \varphi_t(x,t). \qquad (4)$$

Here $u(x,t)$ is an arbitrary function of the class $C^{t_0\tau}$ $(\tau \le T)$. We note that if $u(x,t)$ were a sufficiently smooth solution of problem (1)–(2), then $w = u_t$ would be a solution of problem (3)–(4), and conversely: if, for some function $u(x,t) \in C^{t_0\tau}$, $w = u_t$ is a solution of problem (3)–(4), then $u(x,t)$ is a solution of problem (1)–(2).

The proof of the theorem consists in constructing such a $u \in C^{t_0 t'}$ that u_t satisfies (3) and (4).

On the basis of the results of Friedman' s work (3), problem (3)–(4) is solvable for any function $u \in C^{t_0\tau}$, and the solution $w \in C^{2+\alpha}(Q_{t_0 T})$. Thus problem (3)–(4) defines an operator $w = Mu$, taking $C^{t_0\tau}$ into $C^{2+\alpha}(Q_{t_0\tau})$. We shall construct another operator:

$$v = Nu = u_0(x) + \int_{t_0}^{t} w(x,s)\,ds. \qquad (5)$$

Obviously, this operator takes functions of the class $C^{t_0\tau}$ again into $C^{t_0\tau}$. Lemmas 1–4 are devoted to the construction of a fixed point of the transformation (5), which will complete the proof of the theorem.

For what follows, denote by $K_0(a)$ a common upper bound for the moduli of the function F and of all its derivatives required in condition 1) of the theorem, and also for their Hölder constants with respect to (x,t) in the domain R_a.

Lemma 1. Let $u, u_1, u_2 \in C^{t_0\tau}$, $|u|_{2+\alpha}^{t_0\tau} \le \lambda$, $|u_i|_{2+\alpha}^{t_0\tau} \le \lambda$, $\tau \le T$, $i = 1,2$. Then the following estimates hold:

$$|Mu|_{2+\alpha}^{s\tau} \leq K_2(\lambda), \qquad t_0 \leq s \leq \tau; \tag{6}$$

$$|Mu_1 - Mu_2|_{2+\alpha}^{t_0\tau} \leq K_3(\lambda)|u_1 - u_2|_{2+\alpha}^{t_0\tau}. \tag{7}$$

The functions $K_2(\lambda)$, $K_3(\lambda)$ do not depend on s, τ and are determined by the domain Q_{t_0T}, the conditions (4), and the functions $\nu(\lambda)$, $K_0(\lambda)$.

Proof. For the coefficients of equation (3) it is not difficult to obtain an estimate of the norms in $C^\alpha(Q_{s\tau})$

$$|\,|_{\alpha}^{s\tau} \leq K_0(\lambda)(2+\lambda) = K_1(\lambda). \tag{8}$$

The same estimate is, obviously, also valid for all second derivatives of the functions F required by the theorem. Then, according to (^{3}), taking (8) into account, we obtain the estimate

$$|w|_{2+\alpha}^{s\tau} \leq K(K_1(\lambda), \nu(\lambda))(K_1(\lambda) + |\psi|_{2+\alpha}^{s\tau}). \tag{9}$$

Here ψ is any extension into the domain $Q_{s\tau}$ of the functions $w(x,s)$ and $\varphi_t(x,t)$ $(s \leq t \leq \tau)$; $K(x,y)$ is determined by the domain Q_{t_0T}; the function ψ can be chosen so that $|\psi|_{2+\alpha}^{s\tau} \leq |w|_{2+\alpha}^{t_0T}$, and for the latter one can write an estimate analogous to (9), where instead of $|\psi|_{2+\alpha}^{s\tau}$ one may put $|\psi_2|_{2+\alpha}^{t_0T}$ (see condition 3) of the theorem). Substituting this estimate for $|w|_{2+\alpha}^{t_0T}$ in place of $|\psi|_{2+\alpha}^{sT}$ in (9), we obtain (6).

If, using Hadamard' s device, we write the equation satisfied by the function $Mu_1 - Mu_2$, then inequality (9) for this equation gives the estimate (7).

Lemma 2. The following estimates are valid:

$$|Nu|_{2+\alpha}^{t_0\tau} \leq |Nu|_{2+\alpha}^{t_0t_0} + \left[1 + (\tau - t_0)^{-\alpha/2}\right] \int_{t_0}^{\tau} |Mu|_{2+\alpha}^{t_0s}\, ds; \tag{10}$$

$$|Nu_1 - Nu_2|_{2+\alpha}^{t_0\tau} \leq \left[1 + (\tau - t_0)^{-\alpha/2}\right] \int_{t_0}^{\tau} |Mu_1 - Mu_2|_{2+\alpha}^{t_0s}\, ds. \tag{11}$$

Proof. Consider functions $g \in C^{2+\alpha}(Q_{t_0\tau})$ and

$$f(x,t) = f(x,t_0) + \int_{t_0}^{t} g(x,s)\, ds. \tag{12}$$

Since the derivatives of the function f are also represented through the corresponding derivatives of the function g in the form (12), it is obvious that:

$$|f|_2^{t_0\tau} \leq |f|_2^{t_0t_0} + \int_{t_0}^{\tau} |g|_2^{t_0s}\, ds. \tag{13}$$

For the Hölder constant $H_\alpha^{t_0\tau}(f)$, from the representation (12) it is not difficult to obtain the estimate

$$H_\alpha^{t_0\tau}(f) \leq H_\alpha^G(f(x,t_0)) + \int_{t_0}^{\tau} H_\alpha^{t_0s}(g)\, ds + (\tau - t_0)^{-\alpha/2} \int_{t_0}^{\tau} |g|_0^{t_0s}\, ds.$$

This estimate is also valid for the derivatives of f; therefore, taking (13) into account and using the trivial estimate $|g|_2^{t_0s} \leq |g|_{2+\alpha}^{t_0s}$, we obtain:

$$|f|_{2+\alpha}^{t_0s} \leq |f|_{2+\alpha}^{t_0t_0} + \left[1 + (\tau - t_0)^{-\alpha/2}\right] \int_{t_0}^{\tau} |g|_{2+\alpha}^{t_0s}\, ds. \tag{14}$$

By virtue of (5), estimate (14) holds for Nu and $Nu_1 - Nu_2$, which gives us (10) and (11). Note that $|Nu|_{2+\alpha}^{t_0t_0} = \lambda_0$ does not depend on the particular choice of u, but is determined only by $u_0(x)$ and $w_0(x)$.

Lemma 3. Let $\lambda > \lambda_0$ and $t_1 > t_0$ be such that

$$(t_1 - t_0)^{1-\alpha/2} + (t_1 - t_0) \leqslant \frac{\lambda - \lambda_0}{K_2(\lambda)}. \tag{15}$$

Then in any class $C^{t_0\tau}$ $(\tau \leqslant t_1)$ the operator N maps the set $\{u : |u|_{2+\alpha}^{t_0\tau} \leqslant \lambda\}$ into itself.

Proof follows trivially from (10), (6), and (15). Take $t' = \min\{t_1, t_0{+}1\}$ and an arbitrary function $v_0 \in C^{t_0t'}$ such that $|v_0|_{2+\alpha}^{t_0t'} \leqslant \lambda^*$, and construct the sequence $v_{n+1} = Nv_n$ $(n = 0, 1, \ldots)$.

Lemma 4. The sequence $v_n(x,t)$ converges to some function $u(x,t)$ in the space $C^{2+\alpha}(Q_{t_0t'})$, and $Nu = u$.

Proof. Denote $Mv_n = w_n$. Then

$$v_{n+1}(x,t) = u_0(x,t) + \int_{t_0}^{t} w_n(x,s)\, ds. \tag{16}$$

According to (11), (7), and Lemma 3, we have

$$|v_{n+1} - v_n|_{2+\alpha}^{t_0t} \leqslant K_3(\lambda)\,[1 + (t - t_0)^{-\alpha/2}] \int_{t_0}^{t} |v_n - v_{n-1}|_{2+\alpha}^{t_0s}\, ds, \qquad t \leqslant t'.$$

Hence, by induction, we easily obtain

$$|v_{n+1} - v_n|_{2+\alpha}^{t_0 t} \leqslant 2\lambda \frac{\left[2K_3(\lambda)(t-t_0)^{1-\alpha/2}\right]^n}{n!}.$$

This ensures the convergence of the sequence v_n. Let $u(x,t) = \lim\limits_{n\to\infty} v_n(x,t)$. From (7) there also follows the convergence of the sequence w_n. Let $w(x,t) = \lim\limits_{n\to\infty} w_n(x,t)$. Since one may pass to the limit in equation (3), we have $w = Mu$. Passing to the limit in (16), we obtain $u = Nu$. It remains to note that u and $w = u_t$ belong to $C^{2+\alpha}(Q_{t_0 t'})$. Lemma 4, and thereby the theorem, are proved.

3. In conclusion we note a simple consequence concerning the continuation of the solution $u(x,t)$.

Corollary. For every $\lambda > \lambda_0$ there exists $s > t_0$ such that the solution $u(x,t)$ of problem (1)–(2) exists in the domain $Q_{t_0 s}$ and, together with u_t, belongs to the space $C^{2+\alpha}(Q_{t_0 s})$; moreover either $s = T$, $|u|_{2+\alpha}^{t_0 T} \leqslant \lambda$, or $s < T$, $|u|_{2+\alpha}^{t_0 s} = \lambda$.

Indeed, if $t' < T$ and $|u|_{2+\alpha}^{t_0 t'} = \lambda' < \lambda$, then, taking $u'(x) = u(x,t')$ as the initial condition, we fall under the conditions of the theorem and, consequently, the solution $u(x,t)$ is continued into some domain $Q_{t't''}$, where $(t''-t')^{1-\alpha/2} + (t''-t') \leqslant (\lambda - \lambda')/K_2(\lambda)$ and $t'' - t' \leqslant 1$. If we also observe that, by Lemma 1, the function $K_2(\lambda)$ does not depend on t', then our assertion becomes obvious.

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[*] As v_0 one may take

$$v_0 = u_0 + \int_{t_0}^{t} \psi_2(x,s)\,ds.$$

Since $|\psi_2|_{2+\alpha}^{t_0 T} \leqslant K_2(\lambda)$ for any $\lambda > \lambda_0$ (by the definition of $K_2(\lambda)$), it follows from (14) and (15) that

$$|v_0|_{2+\alpha}^{t_0 t'} \leqslant \lambda.$$

Note: Figure translations are in progress. See original paper for figures.