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1963

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Abstract

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ON THE STABILITY OF HIGHER CORRELATION FUNCTIONS IN A PLASMA

(Presented by Academician L. I. Sedov on 4 V 1963)

In deriving kinetic equations for the first distribution functions F_a in a plasma ([1]), it is assumed that, for the higher correlation functions, the influence of the particular initial data rapidly dies out and they are completely expressed in terms of the first distribution function. This presupposes certain properties of the equations for the higher correlation functions. We shall show that, in the case of first distribution functions that are unstable in the sense of the occurrence of plasma waves, the stationary solutions of the equations for the higher correlations are also unstable. In this paper it is assumed that the magnetic field is absent.

Let us consider the distribution functions $F_{a_1 \dots a_s}(x_1, \dots, x_s)$ in a multicomponent plasma consisting of particles of α species, giving the probability density of the joint distribution of s particles of species $a_1 \dots a_s$ in the phase space $x_1 \dots x_s$, where x_i denotes the totality of the coordinates $\mathbf{q}_i$ and velocities $\mathbf{v}_i$ of a particle of species a_i. The chain of equations for these functions, first obtained by N. N. Bogolyubov ([1]), has the form

$$\frac{\partial}{\partial t} F_{a_1 \dots a_s}(x_1 \dots x_s) + \sum_{i=1}^{s} \mathbf{v}_i \frac{\partial}{\partial \mathbf{q}_i} F_{a_1 \dots a_s} + \sum_{\substack{i,k=1 \\ i \neq k}}^{s} \frac{1}{m_{a_i}} \frac{\partial}{\partial \mathbf{q}_i} U_{a_i a_k}(|\mathbf{q}_i - \mathbf{q}_k|) \frac{\partial}{\partial \mathbf{v}_i} F_{a_1 \dots a_s} = \sum_{i=1}^{s} \sum_{a_{s+1}=1}^{\alpha} \frac{n_{a_{s+1}}}{m_{a_i}} \int \frac{\partial}{\partial \mathbf{q}_i} U_{a_i a_{s+1}}(|\mathbf{q}_i - \mathbf{q}_{s+1}|) \frac{\partial}{\partial \mathbf{v}_i} F_{a_1 \dots a_{s+1}} \, dx_{s+1}, \tag{1}$$

where n_{a_i} denotes the mean density of particles of species a_i.

Write the function $F_{a_1 \dots a_s}$ in the form

$$F_{a_1 \dots a_s} = F^0_{a_1 \dots a_s} + F^1_{a_1 \dots a_s} + \dots + F^{s-1}_{a_1 \dots a_s}, \tag{2}$$

where each function $F^m_{a_1 \dots a_s}$ is represented as the sum of all possible products of $m+1$ functions $F^0_{a_1 \dots a_k}$, corresponding to all possible partitions of $a_1 \dots a_s$ into $m+1$ groups. This relation makes it possible to express $F^0_{a_1 \dots a_s}$ through distribution functions of the s-th and lower orders. From the principle of weakening of correlations, introduced by Bogolyubov, it can be shown that $F^0_{a_1 \dots a_s}$ tends to zero if $|\mathbf{q}_i - \mathbf{q}_j| \to \infty$ for at least one pair i, j. We shall call $F^0_{a_1 \dots a_s}$ irreducible distribution functions.

We shall consider only the spatially homogeneous case. It is easy to see that, in the spatially homogeneous case, $F^0_{a_1 \dots a_s}$ depends on $s-1$ differences $\mathbf{q}_i - \mathbf{q}_j$. Similarly, any term $F^m_{a_1 \dots a_s}$ depends on

$s-1-m$ differences $\mathbf{q}_i - \mathbf{q}_j$. This circumstance makes it possible to write the chain of equations only for the irreducible functions $F^0_{a_1 \dots a_s}$, using equation (1):

$$\frac{\partial}{\partial t} F^0_{a_1 \dots a_s} + \sum_{i=1}^{s} \mathbf{v}_i \frac{\partial}{\partial \mathbf{q}_i} F^0_{a_1 \dots a_s} + \sum_{\substack{i,k=1 \\ i \neq k}}^{s} \frac{1}{m_{a_i}} \frac{\partial}{\partial \mathbf{q}_i} U_{a_i a_k}(|\mathbf{q}_i - \mathbf{q}_k|) \frac{\partial}{\partial \mathbf{v}_i} \left(F^0_{a_1 \dots a_s} + F^1_{a_1 \dots a_s; a_i a_k} \right) =$$

$$= \sum_{i=1}^{s} \sum_{a_{s+1}=1}^{\alpha} \frac{n_{a_{s+1}}}{m_{a_i}} \int \frac{\partial}{\partial \mathbf{q}_i} U_{a_i a_{s+1}}(|\mathbf{q}_i - \mathbf{q}_{s+1}|) \frac{\partial}{\partial \mathbf{v}_i} \left(F^0_{a_1 \dots a_{s+1}} + F^1_{a_1 \dots a_{s+1}; a_i a_{s+1}} \right) d^3 v_{s+1} d^3 q_{s+1}. \tag{3}$$

Here the quantity $F^1_{a_1 \dots a_s; a_i a_k}$ denotes the sum of those terms in the expression for $F^1_{a_1 \dots a_s}$ in which a_i and a_k enter into different factors. The system of equations (3) is equivalent to system (1) and is obtained from equations (1), after substituting there the expressions (2), by selecting the terms with the largest number of arguments.

We shall consider a completely ionized plasma at sufficiently high temperatures, when the plasma parameter $\mu = e^3 n^{1/2} / (\varkappa T)^{3/2} \ll 1$, where e is the electron charge, n is the density of charged particles, T is the absolute temperature, and $\varkappa$ is Boltzmann' s constant.

The quantity $U_{a_i a_j}$ in a completely ionized plasma is the Coulomb potential energy of interaction of two particles of species a_i and a_j. If the F_a are assumed not to depend on time, then the system of equations (3) in this case has stationary solutions $\Phi_{a_1 \dots a_s}$, which are represented by series in powers of μ ([1,3]), and the magnitude of the s-th irreducible correlation is of order μ^{s-1} and has a characteristic correlation length of the order of the Debye radius.

Let us consider a nonstationary deviation of the solution $F^0_{a_1 \dots a_s}$ from $\Phi_{a_1 \dots a_s}$

$$F^0_{a_1 \dots a_s} = \Phi_{a_1 \dots a_s} + f^0_{a_1 \dots a_s}$$

over times shorter than the characteristic time of variation of F_a, so that F_a may be considered independent of time.

Let us compare the orders of magnitude of the different terms in equation (3). It is easy to see that

$$\sum_{i,k,\ i\neq k}^{s} \frac{1}{m_{a_i}} \frac{\partial}{\partial \mathbf{q}_i} U_{a_i a_k}(|\mathbf{q}_i - \mathbf{q}_k|) \frac{\partial}{\partial \mathbf{v}_i} f^0_{a_1 \ldots a_s}$$

has an extra factor μ compared with the term

$$\sum_{i=1}^{s} \sum_{a_{s+1}=1}^{\alpha} \frac{n_{a_{s+1}}}{m_{a_i}} \int \frac{\partial}{\partial \mathbf{q}_i} U_{a_i a_{s+1}}(|\mathbf{q}_i - \mathbf{q}_{s+1}|) \frac{\partial F^0_{a_i}}{\partial \mathbf{v}_i} f^0_{a_1 \ldots a_{i-1} a_{i+1} \ldots a_{s+1}}\, dx_{s+1},$$

and we shall neglect this quantity.

Suppose that we have the following initial data:

$$f^0_{a_1 \ldots a_s}\big|_{t=0} \neq 0, \qquad f_{a_1 \ldots a_{s'}}\big|_{t=0} = 0 \quad \text{for } s' \neq s. \tag{4}$$

We shall assume the quantity $f^0_{a_1 \ldots a_s}\big|_{t=0}$ to be small, of the order of some parameter ε. According to equations (3), the functions $F^0_{a_1 \ldots a_{s'}}$ for $s' < s$ will be of order ε at subsequent times. Hence, by virtue of the same

it follows from the equations that all $f^0_{a_1 \ldots a_{s+1}}$ at subsequent moments of time will be of order either ε^2, or $\varepsilon\mu$.

Returning to the equation for $f^0_{a_1 \ldots a_s}$, we see that the terms obtained from $F^1_{a_1 \ldots a_s}$, $F^1_{a_1 \ldots a_{s+1}}$, and $F^0_{a_1 \ldots a_{s+1}}$ have orders ε^2 and $\varepsilon\mu^b$, $b \geqslant 1$.

In studying stability, nonlinear terms of order ε^2 may be neglected; analogously, we shall neglect terms containing an extra power of the parameter μ. It is clear that the case of general initial data is obtained by superposition of the initial data written above.

Thus, the equation for determining $f^0_{a_1 \ldots a_s}$ has the form

$$\frac{\partial}{\partial t} f^0_{a_1 \ldots a_s} + \sum_{i=1}^{s} \left(\mathbf{v}_i \frac{\partial}{\partial \mathbf{q}_i} \right) f^0_{a_1 \ldots a_s} + \sum_{i=1}^{s} \sum_{a_{s+1}=1}^{\alpha} \frac{n_{a_{s+1}}}{m_{a_i}} \frac{\partial F_{a_i}}{\partial \mathbf{v}_i} \int e_{a_i} e_{a_{s+1}} \frac{(\mathbf{q}_i - \mathbf{q}_{s+1})}{|\mathbf{q}_i - \mathbf{q}_{s+1}|^3} f^0_{a_1 \ldots a_{i-1} a_{i+1} \ldots a_{s+1}}\, d^3 v_{s+1}\, d^3 q_{s+1}. \tag{5}$$

Introduce the vector notation $\mathbf{f}^0_s$ for the collection of all $f^0_{a_1 \ldots a_s}$. Then the last equation can be written in the form

$$\frac{\partial}{\partial t}\mathbf{f}_s^0 + \sum_{i=1}^{s} \hat{L}_i \mathbf{f}_s^0 = 0,$$

where $\hat{L}_i$ is an integro-differential operator coinciding with the integro-differential operator in the linearized Vlasov equation for a perturbation of the first distribution function

$$\frac{\partial f_a}{\partial t} + \sum_b \delta_{ab}\, \mathbf{v}_b \frac{\partial f_b}{\partial \mathbf{q}_b} + \sum_b \frac{n_b}{m_a} e_a e_b \frac{\partial F_a}{\partial \mathbf{v}_a} \int \frac{(\mathbf{q}_a - \mathbf{q}_b)}{|\mathbf{q}_a - \mathbf{q}_b|^3} f_b\, d^3 v_b d^3 q_b = 0$$

or, in vector form,

$$\frac{\partial \mathbf{f}_1}{\partial t} + \hat{L}\mathbf{f}_1 = 0. \tag{6}$$

It is easy to see that the product of the solutions of equations (6) $f_{a_1} \cdots f_{a_s}$ is a solution of equation (5). Therefore, if equation (6) has solutions that grow without bound with time, then equation (5) also has solutions that grow without bound with time.

For the explicit determination of solutions of equations (5), let us carry out the Fourier transform with respect to all $\mathbf{q}_i$:

$$\int \mathbf{f}_s^0(\mathbf{k}_1 \ldots \mathbf{k}_s, \mathbf{v}_1 \ldots \mathbf{v}_s, t)\, \delta(\mathbf{k}_1 + \ldots + \mathbf{k}_e)\, e^{i\mathbf{k}_1\mathbf{q}_1 + \ldots + i\mathbf{k}_s\mathbf{q}_s}\, d^3k_1 \ldots d^3k_s =$$

$$= \mathbf{f}_s^0(\mathbf{q}_1 \ldots \mathbf{q}_s, \mathbf{v}_1 \ldots \mathbf{v}_s, t).$$

The general solution for $\mathbf{f}_s^0(\mathbf{k}_1 \ldots \mathbf{k}_s, \mathbf{v}_1 \ldots \mathbf{v}_s, t)$ with arbitrary initial data can be written by means of the matrix Green function $\hat{G}_s$ in the form

$$\mathbf{f}_s^0(\mathbf{k}_1 \ldots \mathbf{k}_s, \mathbf{v}_1 \ldots \mathbf{v}_s, t) =$$

$$= \int \hat{G}_s(t, \mathbf{k}_1 \ldots \mathbf{k}_s, \mathbf{v}_1 \ldots \mathbf{v}_s, \mathbf{v}_1' \ldots \mathbf{v}_s') \mathbf{f}_s^0(\mathbf{k}_1 \ldots \mathbf{k}_s, \mathbf{v}_1' \ldots \mathbf{v}_s', 0)$$

$$\times\, d^3v_1' \ldots d^3v_s'. \tag{7}$$

The matrix Green’ s function $\hat{G}_s$ is the product of the corresponding matrix Green’ s functions $\hat{G}_1(t, \mathbf{k}, \mathbf{v})$ for system (6):

$$\hat{G}_s = \prod_{i=1}^{s} \hat{G}_1(t, \mathbf{k}_i, \mathbf{v}_i, \mathbf{v}_i').$$

This follows immediately from the fact that the product of solutions of equation (6) is a solution of equation (5). It follows from this representation that all properties of the solutions of equation (6) are transferred to the solutions of equation (5). Hence it follows that, in the case of instability, the fastest-growing part of the function $f^0_{a_1 \dots a_s}(\mathbf{k}_1 \dots \mathbf{k}_s, \mathbf{v}_1 \dots \mathbf{v}_s, t)$ is proportional to

$$\prod_{i=1}^{s} \varphi_{a_i}(\mathbf{k}_i, \mathbf{v}_i, t), \tag{8}$$

where φ_a are the growing solutions of the Vlasov equation (6), which, as is known, have the form (see ([2]))

$$\varphi_a = \frac{e_a}{m_a} \frac{i\mathbf{k}}{k^2} \frac{1}{-i\Omega_k + i\mathbf{kv}} \frac{\partial F_a}{\partial \mathbf{v}} e^{-i\Omega_k t},$$

where Ω_k is the root of the dispersion equation with the largest imaginary part

$$\mathcal{E}^{(+)}(k, \Omega) \equiv 1 - \sum_a \frac{4\pi e_a}{m_a k^2} i\mathbf{k} \int \frac{\partial F_a}{\partial \mathbf{v}} \frac{d^3 v}{-i\Omega + i\mathbf{kv}} = 0. \tag{9}$$

In the case of stability only the quantity $R = \int f_s \, d^3 v_1 \dots d^3 v_s$ decays (see ([2])). If the function $f_s|_{t=0}$ can be analytically continued into a sufficiently wide strip $0 \geq \operatorname{Im} v_j^\alpha \geq h$ in all components of the vectors $\mathbf{v}_j$, then the damping will proceed according to the law $\exp(s \operatorname{Im} \Omega_k t)$, where Ω_k is the root of equation (9) with the largest imaginary part $\operatorname{Im} \Omega_k < 0$. Otherwise, the damping law of R depends essentially on the analytic properties of the initial data ([2]).

In investigating the solutions of equation (5) it was assumed that the functions F_a do not depend on time. Since the rate of change of the irreducible distribution functions is of order $\operatorname{Im} \Omega_k$, not directly related to μ, while the rate of change of the first distribution function is of order μ, then for small μ the first distribution function can indeed be regarded as constant during the time interval over which the functions $f^0_{a_1 \dots a_s}$ $(s > 1)$ have time to grow appreciably.

Thus, in the presence of roots (8) with $\operatorname{Im} \Omega_k > 0$, the stationary solutions for all irreducible distribution functions are unstable and, consequently, in this case they cannot be regarded as depending only on the first distribution functions.

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Received
24 IV 1963

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Note: Figure translations are in progress. See original paper for figures.