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Abstract

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MATHEMATICS

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A UNIVERSAL COEFFICIENT THEOREM FOR SPECTRAL COHOMOLOGY GROUPS OF DIFFERENTIAL GROUPS

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1. Let I be a certain set, and let $J \subseteq I \times I$; further, let A_α, where $\alpha \in I$, be additive abelian groups, and let $\eta^\alpha_\beta : A_\alpha \to A_\beta$ be homomorphisms defined for $(\alpha, \beta) \in J$. By $\mathcal{C}_{I,J}$ we shall denote the category whose objects are systems $\{A_\alpha; \eta^\alpha_\beta\}$, and whose mappings

$$\varphi : \{A_\alpha; \eta^\alpha_\beta\} \to \{A'_\alpha; \eta'^\alpha_\beta\}$$

are such systems $\{\varphi_\alpha\}$ of homomorphisms $\varphi_\alpha : A_\alpha \to A'_\alpha$ for which

$$\varphi_\beta \circ \eta^\alpha_\beta = \eta'^\alpha_\beta \circ \varphi_\alpha$$

(note that the objects of the category $\mathcal{C}_{I,J}$ will be mutually compatible systems of groups in the terminology proposed by us in (3)).

Now let $\mathfrak{N}$ be the set of nonnegative integers, and let $\mathfrak{M} \subseteq \mathfrak{N} \times \mathfrak{N}$ consist of the pairs $(0, m)$, (m', m), and (m, m'), where m and m' are all possible natural numbers for which $m \mid m'$. We shall consider the category $\mathcal{C} = \mathcal{C}_{\mathfrak{N},\mathfrak{M}}$. In accordance with what was said above, the objects of this category will be systems

$$\{A_0, A_m; \eta^0_m, \eta^{m'}_m, \eta^m_{m'}\}.$$

2. Let G be an abelian group, and let L be a differential abelian group with differential operator $d : L \to L$, $dd = 0$. The cohomology group $H_*(L, G)$ of the group L with coefficient group G is the homology group

$$H(\mathrm{Hom}(L, G))$$

(we put the asterisk below, since in the case of a graded group L the differential operator d is assumed to have degree $+1$, and not -1*). Consider the totality of all finit generated differential subgroups (i.e., subgroups closed with respect to the operation d) L_α of the group L. They are partially ordered by inclusion, i.e. $\alpha \prec \beta$ means that $L_\alpha \subseteq L_\beta$, and they form a direct spectrum (directed set) $\{L_\alpha; i^\alpha_\beta\}$, where

$$i^\alpha_\beta : L_\alpha \to L_\beta$$

are the inclusion mappings. Obviously,

$$\lim_{\to}\{L_\alpha; i^\alpha_\beta\} = L.$$

The mappings i^α_β induce mappings of cohomology groups

$$i^\beta_{*\alpha} : H_*(L_\beta, G) \to H_*(L_\alpha, G);$$

the limit of the inverse spectrum

$$\{H_*(L_\alpha, G); i^\beta_{*\alpha}\}$$

will be called the **spectral cohomology group** of the differential group L with coefficient group G, and will be denoted by $\mathscr{H}(L, G)$, so that

$$\mathscr{H}(L, G) = \lim_{\leftarrow}\{H_*(L_\alpha, G); i^\beta_{*\alpha}\} = \lim_{\leftarrow}\{H(\mathrm{Hom}(L_\alpha, G)); i^\beta_{*\alpha}\}.$$

Obviously, $\mathscr{H}$ is a functor, contravariant in the first argument and covariant in the second.

3. The functor $\mathscr{H}$, in contrast to H_* (see, for example, (^{4}), 9.1* and the remark to 9.3*), commutes with taking the limit of the corresponding spectrum. Namely, if L is the limit of a direct spectrum of differential groups

$$L = \lim_{\to}\{L^\lambda; \sigma^\lambda_\mu\}, \qquad \sigma^\lambda_\mu : L^\lambda \to L^\mu \quad (\lambda \prec \mu),$$

then $\mathscr{H}(L, G)$ will be the limit of the inverse spectrum of their spectral cohomology groups

$$\mathscr{H}(L, G) = \lim_{\leftarrow}\{\mathscr{H}(L^\lambda, G); \sigma^\mu_{*\lambda}\},$$

where $\sigma^\mu_{*\lambda} : \mathscr{H}(L^\mu, G) \to \mathscr{H}(L^\lambda, G)$ is the mapping naturally induced by the homomorphism $\sigma^\lambda_\mu : L^\lambda \to L^\mu$ $(\lambda < \mu)$.

Indeed, let $u \in \mathscr{H}(L, G)$, i.e. $u = \{u_\alpha\}$, $u_\alpha \in H_*(L_\alpha, G) = H(\mathrm{Hom}(L_\alpha, G))$ (L_α, as before, ranges over all finitely generated differential subgroups of L; the same meaning will be attached below to an index placed below a differential group). Let y_α be cycles from the homology classes u_α, i.e. $y_\alpha : L_\alpha \to G$, with $\partial(y_\alpha) = y_\alpha \circ d = 0$, where d is the differential operator in the group $\mathrm{Hom}(L, G)$. Denote by $\sigma^\lambda : L^\lambda \to L$ the limiting homomorphisms of the direct spectrum $\{L^\lambda; \sigma^\lambda_\mu\}$, and let $\sigma^\lambda(L^\lambda_\gamma) = L_\alpha$, where L^λ_γ are all possible finitely

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* An analogous remark also applies to our previous papers (^{2},^{3}).

generated differential subgroups of the group L^λ, so that the index α is a function $\alpha = (\lambda, \gamma)$ of the indices λ and γ. Define mappings $y^\lambda_\gamma : L^\lambda_\gamma \to G$ by setting $y^\lambda_\gamma = y_\alpha \circ \sigma^\lambda$, $\alpha = (\lambda, \gamma)$ (more precisely, one should write $y^\lambda_\gamma = y_\alpha \circ (i^\alpha)^{-1} \circ \sigma^\lambda$, where i^α are the inclusion mappings $L_\alpha \to L$). We have

$$\partial(y^\lambda_\gamma) = y^\lambda_\gamma \circ d = y_\alpha \circ \sigma^\lambda \circ d = y_\alpha \circ d \circ \sigma^\lambda = \partial(y_\alpha) \circ \sigma^\lambda = 0,$$

i.e. y^λ_γ will be a cycle; its homology class u^λ_γ does not depend on the choice of the cycles y_α in the homology classes u_α, for if $y_\alpha \sim 0$, i.e. $y_\alpha = \partial(x_\alpha) = x_\alpha \circ d$ $(x_\alpha : L_\alpha \to G)$, then

$$y^\lambda_\gamma = y_\alpha \circ \sigma^\lambda = x_\alpha \circ d \circ \sigma^\lambda = x_\alpha \circ \sigma^\lambda \circ d = \partial(x_\alpha \circ \sigma^\lambda) \sim 0.$$

Further, it is easily checked that, for fixed λ, $u^\lambda = \{u^\lambda_\gamma\}$ is an element of the group $\mathcal{H}(L^\lambda, G)$ and that $\{u^\lambda\}$ is an element of the group

$$\varprojlim \{\mathcal{H}(L^\lambda, G); \sigma^\mu_{*\lambda}\}.$$

Thus a natural homomorphism

$$\Phi : \mathcal{H}(L, G) \to \varprojlim \{\mathcal{H}(L^\lambda, G); \sigma^\mu_{*\lambda}\}$$

is defined.

It is easy to see that it is an isomorphism of these two groups. Indeed, if $\{u^\lambda\}$ is an element of the inverse limit standing on the right, i.e. $u^\lambda = \{u^\lambda_\gamma\} \in \mathcal{H}(L^\lambda, G)$, $\sigma^\mu_{*\lambda}(u^\mu) = u^\lambda$, then there exists its preimage in $\mathcal{H}(L, G)$. In fact, let L_α be an arbitrary finitely generated differential subgroup of the group L, and let $l_1, \dots, l_n$ be the elements generating it. Since L is the direct limit of the groups L^λ, there exists a value of λ for which in the group L^λ there are preimages $l^\lambda_1, \dots, l^\lambda_n$ of these elements, and the differential subgroup L^λ_γ of the group L^λ, generated by the elements $l^\lambda_1, \dots, l^\lambda_n$, is isomorphic to the group L_α, and this isomorphism is effected by the mapping σ^λ. Therefore one can define a mapping $y_\alpha : L_\alpha \to G$ by setting

$$y_\alpha = y^\lambda_\gamma \circ (\sigma^\lambda | L^\lambda_\gamma)^{-1},$$

where $y^\lambda_\gamma : L^\lambda_\gamma \to G$ are cycles from the homology classes u^λ_γ. It is checked directly that y_α will be a cycle, i.e. $\partial(y_\alpha) = y_\alpha \circ d = 0$, and that if $L_\alpha \subseteq L_\beta$, then $i^\beta_\alpha y_\alpha \sim y_\beta$, so that, putting $u = \{u_\alpha\}$, where u_α are the homology classes of the cycles y_α, we obtain an element of the group $\mathcal{H}(L, G)$ for which, as is not hard to see, $\Phi(u) = \{u^\lambda\}$. Similarly, i.e. using again the fact that a finitely generated subgroup of the limit of a direct spectrum of groups always has an isomorphic preimage in one of the groups of this spectrum, it is shown that if $\Phi(u) = 0$, then also $u = 0$.

4. As we have already noted in (³) (Addendum II), the Čech (based on finite covers) cohomology group $H^{(c)}(X)$ of a topological space X (and even the cohomology group $H^{(c)}(X, G)$ with coefficients in a given group G) can,

using only canonical (Kurosh) closed covers or else multiplicative covers with exact refinements (Alexandroff), be regarded as the homology group $H(L)$ of a differential group $L = \varinjlim L_\alpha$, where L_α is the chain group of these covers, which we shall call the chain group of the space X. The usual definition of the Čech cohomology groups $H_{(c)}(X, G)$ means that

$$H_{(c)}(X, G) = \varprojlim H_*(L_\alpha, G).$$

Since L_α are finitely generated groups,

then $H_*(L_*, G) = \mathscr{H}(L_\alpha, G)$, and therefore, according to what was said above,

$$H_{(c)}(X, G) = \varprojlim \mathscr{H}(L_\alpha, G) = \mathscr{H}(L, G),$$

i.e. the Čech homology group of a space is the spectral cohomology group of the group of its chains (with respect to the same coefficient group). Therefore the theorems proved below, concerning spectral cohomology groups, will also be valid for the Čech homology groups of topological spaces. The same is also true for relative Čech homology groups, as well as for the internal homology groups of a locally bicompact space (in the sense of P. S. Aleksandrov).

5. Main assertion*. *If L is a torsion-free differential group, then there is a natural isomorphism:*

$$\mathscr{H}(L, G) = \mathrm{Hom}(\{H(L),\, H_m(L);\ \pi^0_m,\ \pi^{m'}_m,\ \omega^m_{m'}\}, \{G,\, G_m;\ \varphi^0_m,\ \varphi^{m'}_m,\ \psi^m_{m'}\}).$$

Here, as in our previous works ($^{1-3}$), $H_m(L)$ are the homology groups modulo m, i.e. $H_m(L) = H(L \otimes Z_m)$, $G_m = G/mG = G \otimes Z_m$, and

$$\pi^0_m : H(L) \to H_m(L), \quad \pi^{m'}_m : H_{m'}(L) \to H_m(L), \quad \omega^m_{m'} : H_m(L) \to H_{m'}(L)$$

and, respectively,

$$\varphi^0_m : G \to G_m, \quad \varphi^{m'}_m : G_{m'} \to G_m, \quad \psi^m_{m'} : G_m \to G_{m'}$$

are homomorphisms induced by the natural mappings $Z \to Z_m$, $Z_{m'} \to Z_m$, $Z_m \to Z_{m'}$ (m/m'); the symbol Hom refers to the category $\mathscr{C}$.

Proof. Let $u = \{u_\alpha\} \in \mathscr{H}(L, G)$ $(u_\alpha \in H(\mathrm{Hom}(L_\alpha, G)))$. To this element we associate mappings

$$\tilde{u}_0 : H(L) \to G$$

and

$$\tilde{u}_m : H_m(L) \to G_m.$$

Namely, if $h \in H(L)$, $z \in h$, L_α is some finitely generated differential subgroup of L containing z, and $y_\alpha : L_\alpha \to G$ is a cycle from u_α (so that $y_\alpha(dL_\alpha) = 0$), then we put $\tilde{u}_0(h) = y_\alpha(z)$. If $z' \in L_\beta$ and also $\in h$, and $y'_\beta \in u_\beta$, then $z' = z + dl$, $l \in L_\gamma$ $(\gamma > \alpha,\ \gamma > \beta)$ and, consequently,

$$y'_\beta(z') = y''_\gamma(z') + (\partial x_\gamma)(z') = y''_\gamma(z') + x_\gamma(dz') = y''_\gamma(z) = y''_\gamma(z) + y''_\gamma(dl) = y''_\gamma(z) = y_\alpha(z) + (\partial x'_\gamma)(z) = y_\alpha(z) + x'_\gamma($$

for $dz = dz' = 0$ $(y''_\gamma \in u_\gamma;\ x_\gamma, x'_\gamma : L_\gamma \to G)$, so that $\tilde{u}_0$ does not depend on the accidents of the construction. Similarly, if $h_m \in H_m(L) = H(L \otimes Z_m)$, $z_m \otimes 1_m \in h_m$, and $z_m \in L_\alpha$, $y_\alpha \in u_\alpha$, then we put

$$\tilde{u}_m(h_m) = y_\alpha(z_m) \otimes 1_m$$

(1_m is the unit in Z_m), and in the same way we show that $\tilde{u}_m$ is thereby determined uniquely. It is easy to show that, for the constructed mapping $\Psi(u) = \{\tilde{u}_0, \tilde{u}_m\}$, one has

$$\Psi(u) \in \operatorname{Hom}(\{H_n(L);\ \pi, \omega\}, \{G_n;\ \varphi, \psi\}) \quad (n = 0, m),$$

i.e.

$$\tilde{u}_m \circ \pi^0_m = \varphi^0_m \circ \tilde{u}_0, \quad \tilde{u}_m \circ \pi^{m'}_m = \varphi^{m'}_m \circ \tilde{u}_{m'}, \quad \tilde{u}_{m'} \circ \omega^m_{m'} = \psi^m_{m'} \circ \tilde{u}_m \quad (m/m').$$

Ψ is an isomorphism of both groups. Indeed, if $\{\tilde{u}_n\}$ is a system of mappings

$$\tilde{u}_0 : H(L) \to G, \quad \tilde{u}_m : H_m(L) \to G_m$$

(where $\tilde{u}_m \circ \pi^0_m = \varphi^0_m \circ \tilde{u}_0$, etc.), then there exists an element $u = \{u_\alpha\} \in \mathcal{H}(L, G)$, $u_\alpha \in H(\operatorname{Hom}(L_\alpha, G))$, for which $\Psi(u) = \{\tilde{u}_n\}$. In fact, for each finitely generated differential subgroup $L_\alpha \subseteq L$ the free generators a_i in L_α can, as is known, be chosen so that $da_i = m_i a_{i+N}$ for $i \le N$, $da_i = 0$ for $i > N$, where N is some number and the m_i are natural numbers. We now take for u_α the homology class of the cycle $y_\alpha : L_\alpha \to G$, defined as follows: $y_\alpha(a_i) = \tilde{u}_0(v_i)$ for $i > N$, where v_i is the homology class of the cycle a_i, and for $i \le N$ we choose $y_\alpha(a_i)$ so that

$$y_\alpha(a_i) \otimes 1_m = \tilde{u}_{m_i}(v_i^{(m_i)}),$$

where $v_i^{(m_i)}$ is the homology class of the cycle modulo m_i $a_i \otimes 1_{m_i}$. It is easily shown that the system $\{u_\alpha\}$ thus constructed is the desired one. Using the same considerations, we show that from $\Psi(u) = 0$ it follows that $u = 0$, analogously to the way a fact of this kind was proved in our work (3).

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* Using the term proposed in (6), it may be called the functorial theorem on universal coefficients for spectral cohomology groups.

6. In our papers (1–3) it was shown that there exists an isomorphism (which is not, however, natural) of the group $H_m(L) = H(L \otimes Z_m)$ with the direct sum of groups $H_m(L) \approx [H(L)]_m + {}_m[H(L)]$, where ${}_mG$ $(G = H(L))$ is the subgroup of elements g of the group G for which $mg = 0$, under which $\pi_m^0 = (\varphi_m^0, 0)$, $\pi_m^{m'} = (\varphi_m^{m'}, j_m^{m'})$, $\omega_{m'}^m = (\psi_{m'}^m, i_{m'}^m)$ $(m \mid m')$, where the homomorphisms $\varphi_m^0, \varphi_m^{m'}, \psi_{m'}^m$ defined above now refer to the group $G = H(L)$, $i_{m'}^m : {}_mG \to {}_{m'}G$ is the inclusion of subgroups, and $j_m^{m'} : {}_{m'}G \to {}_mG$ is obtained as a result of multiplication by the number $\dfrac{m'}{m}$.

Therefore our preceding assertion gives

$$\mathcal{H}(L, G) \approx \operatorname{Hom}(\{H(L), [H(L)]_m; \varphi_m^0, \varphi_m^{m'}, \psi_{m'}^m\}, \{G, G_m; \varphi_m^0, \varphi_m^{m'}, \psi_{m'}^m\}) +$$

$$+ \operatorname{Hom}(\{0, {}_m[H(L)]; 0, j_m^{m'}, i_{m'}^m\}, \{G, G_m; \varphi_m^0, \varphi_m^{m'}, \psi_{m'}^m\}).$$

It is easy to see that the first direct summand is $\operatorname{Hom}(H(L), G)$, so that, in particular, the sequence with the natural mapping $\mathcal{H}(L, G) \to \operatorname{Hom}(H(L), G) \to 0$ is exact, while the second summand, i.e. the group $\operatorname{Hom}(\{{}_m[H(L)]; j_m^{m'}, i_{m'}^m\}, \{G_m; \varphi_m^{m'}, \psi_{m'}^m\})$ (relative to the category $\mathcal{C}_{\mathfrak{m}\backslash 0 \times \mathfrak{m}}$), is $\mathcal{E}xt(H(L), G)$, where $\mathcal{E}xt(H, G) = \lim_{\leftarrow} \operatorname{Ext}(H_\alpha, G)$ (H_α are all possible finitely generated subgroups of the group H) is the functor introduced under the notation Ext^* in paper (5) (more precisely, $\mathcal{E}xt$ is Ext^* applied to the torsion subgroup of the group H). Indeed, since the finitely generated subgroups L_α of the group L are free, we have exact sequences $0 \to \operatorname{Ext}(H(L_\alpha), G) \to H_*(L_\alpha, G) \to \operatorname{Hom}(H(L_\alpha), G) \to 0$ ((4), 3.3a). By virtue of the left exactness of the inverse-limit functor and the commutation, proved in (5) (Theorem 24.2), of the functor $\mathcal{E}xt$ (and, as is known, Hom, see (4), 9.1) with the corresponding limit, one may (keeping in mind that $L = \lim_{\rightarrow} L_\alpha$) pass in them to the limit, which gives the exact sequence $0 \to \mathcal{E}xt(H(L), G) \to \mathcal{H}(L, G) \to \operatorname{Hom}(H(L), G)$ (as noted above, at the end one may append $\to 0$), showing that the functor under consideration is indeed $\mathcal{E}xt$.

Thus we obtain the desired universal coefficient theorem for a differential group without torsion, both in the form of an exact sequence with natural mappings and in the form of a decomposition into the direct sum $\mathcal{H}(L, G) \approx \operatorname{Hom}(H(L), G) + \mathcal{E}xt(H(L), G)$. For the Čech homology groups of a topological space this frees Theorem 44.1 of paper (5) (with replacement of (44.2) by (40.1)) from all restrictions present there (in the case of discrete coefficient groups).

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