E. M. SEMENOV

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Abstract

Full Text

E. M. SEMENOV

ON A SCALE OF SPACES WITH THE INTERPOLATION PROPERTY

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In ([1]) a maximal scale of spaces connecting the spaces L_1 and L_∞ was constructed. These spaces were first introduced by G. G. Lorentz ([2]). In his notation the Banach space $\Lambda(\alpha)$ $(0 < \alpha < 1)$ consists of all functions measurable on $[0, 1]$ for which

$$\|x\|_{\Lambda(\alpha)} = \alpha \int_0^1 x^*(t) t^{\alpha-1}\, dt < \infty, \tag{1}$$

where $x^*(t)$ denotes the nonincreasing function equimeasurable with $|x(t)|$. The spaces conjugate to the spaces $\Lambda(\alpha)$ are the spaces $M(\alpha)$, consisting of all functions measurable on $[0, 1]$ for which

$$\|x\|_{M(\alpha)} = \sup_{0<h\le 1} \frac{\int_0^h x^*(t)\, dt}{h^\alpha} < \infty. \tag{2}$$

In ([2]) it is shown that the spaces $\Lambda(\alpha)$ are nonreflexive. In the present note a scale of spaces $M_0(\alpha)$ will be constructed, to which $\Lambda(\alpha)$ are conjugate, and interpolation theorems for this scale will be proved.

1. Let $M_0(\alpha)$ denote the set of functions in $M(\alpha)$ for which

$$\lim_{h\to 0} \frac{\int_0^h x^*(t)\, dt}{h^\alpha} = 0. \tag{3}$$

It is easy to verify that every bounded function belongs to $M_0(\alpha)$ and, moreover,

$$M_0(\alpha) \subset L_{\frac{1}{1-\alpha}}. \tag{4}$$

Lemma 1. $M_0(\alpha)$ coincides with the closure of the set of bounded functions in the norm of the space $M(\alpha)$.

Proof. Let $x_N(t)$ denote the truncation of the function $x(t) \in M(\alpha)$. Then

$$\|x - x_N\|_{M(\alpha)} = \sup_{0<h\leq 1} \frac{\int_0^h (x - x_N)^* \, dt}{h^\alpha} =$$

$$= \sup_{0<h\leq mE(|x|>N)} \frac{\int_0^h (x - N \operatorname{sign} x)^* \, dt}{h^\alpha} < \sup_{0<h\leq mE(|x|>N)} \frac{\int_0^h x^* \, dt}{h^\alpha}.$$

Since $\lim_{N\to\infty} mE(|x| > N) = 0$, it follows by virtue of (3) that $\lim_{N\to\infty} \|x - x_N\|_{M(\alpha)} = 0$.

On the other hand, if $x(t) \in M(\alpha)$ and $x(t) \notin M_0(\alpha)$, then there exists a sequence $h_k \downarrow 0$ such that

$$\lim_{k\to\infty} \int_0^{h_k} x^*(t) \, dt \, / \, h_k^\alpha = \lambda > 0.$$

Then for any $y(t) \in M_0(\alpha)$

$$\|x - y\|_{M(\alpha)} = \sup_{0<h\leq 1} \frac{\int_0^h (x - y)^* \, dt}{h^\alpha} \geq \lim_{k\to\infty} \frac{\int_0^{h_k} (x - y)^* \, dt}{h_k^\alpha} \geq$$

$$\geq \lim_{k\to\infty} \left(\frac{\displaystyle\int_0^{h_k} x^*(t) \, dt}{h_k^\alpha} - \frac{\displaystyle\int_0^{h_k} y^*(t) \, dt}{h_k^\alpha} \right) = \lim_{k\to\infty} \frac{\displaystyle\int_0^{h_k} x^*(t) \, dt}{h_k^\alpha} = \lambda.$$

The lemma is proved.

$M_0(\alpha)$, being a closed subspace of $M(\alpha)$, is a complete normed space. If $\chi_E(t)$ is the characteristic function of a measurable set $E \subset [0, 1]$, then

$$\|\chi_E(t)\|_{M(\alpha)} = \sup_{0<h\leq 1} \frac{\displaystyle\int_0^h \chi_E^*(t) \, dt}{h^\alpha} = \sup_{0<h\leq mE} \frac{\displaystyle\int_0^h 1 \cdot dt}{h^\alpha} = mE^{1-\alpha}. \tag{5}$$

Theorem 1. *The formula*

$$f(x) = \int_0^1 x(t)y(t) \, dt, \qquad \text{where } y(t) \in \Lambda(\alpha), \tag{6}$$

gives the general form of a linear functional on $M_0(\alpha)$, and moreover $\|f\| = \|y\|_{\Lambda(\alpha)}$.

Proof. As shown in ([2]),

$$|f(x)| \leq \|x\|_{M(\alpha)} \|y\|_{\Lambda(\alpha)},$$

therefore $f(x)$ is defined on the whole space $M_0(\alpha)$ and $\|f\| \leq \|y\|_{\Lambda(\alpha)}$.

Let us prove that equality holds. It is clear that for every function $y(t) \in \Lambda(\alpha)$ there exists a function $\theta(t)$ such that $\theta^*(t) = \alpha t^{\alpha-1}$,

$$\int_0^1 \theta(t) y(t)\, dt = \int_0^1 \alpha t^{\alpha-1} y^*(t)\, dt.$$

Denote by $\theta_N(t)$ the truncation of the function $\theta(t)$ and

$$a_N = (\alpha/N)^{1/(1-\alpha)}.$$

Then $\|\theta_N\|_{M(\alpha)} \leq \|\theta\|_{M(\alpha)} = 1$, $\theta_N(t) \in M_0(\alpha)$, and

$$f(\theta_N) = \int_0^1 \theta_N(t) y(t)\, dt = N \int_0^{a_N} y^*(t)\, dt + \int_{a_N}^1 \alpha t^{\alpha-1} y^*(t)\, dt \geq$$

$$\geq \alpha \int_{a_N}^1 y^*(t) t^{\alpha-1}\, dt = \alpha \int_0^1 y^*(t) t^{\alpha-1}\, dt - \alpha \int_0^{a_N} y^*(t) t^{\alpha-1}\, dt.$$

Since the function $y^*(t)t^{\alpha-1}$ is summable on $[0, 1]$, it follows that

$$\sup_N f(\theta_N) \geq \lim_{N\to\infty} \left(\alpha \int_0^1 y^*(t) t^{\alpha-1}\, dt - \alpha \int_0^{a_N} y^*(t) t^{\alpha-1}\, dt \right) = \|y\|_{\Lambda(\alpha)}.$$

We shall show that every linear functional on $M_0(\alpha)$ is representable in the form (6). Let $f(x)$ be an arbitrary linear functional on $M_0(\alpha)$. Then, by virtue of (5),

$$|f(\chi_E)| \leq \|f\|\, \|\chi_E\|_{M(\alpha)} = \|f\|\, mE^{1-\alpha}.$$

Therefore $f(\chi_E)$, considered as a function of subsets of the interval $[0, 1]$, is absolutely continuous and, by the Radon-Nikodym theorem, is representable in the form

$$f(\chi_E) = \int_E y(t)\, dt, \tag{7}$$

where $y(t)$ is a summable function on $[0,1]$. It follows from (7) that for every step function $x(t)$ the equality (6) is valid.

Now let $x(t) \in M(\alpha)$. One can construct a sequence of step functions $z_N(t)$ such that $z_N(t)$ converge to $x(t)$ almost everywhere and $|z_N(t)| \leq |x(t)|$. Then $|z_N(t)y(t)|$ will converge almost everywhere to the function $|x(t)y(t)|$, and the inequality

$$\|z_N\|_{M(\alpha)} \leq \|x\|_{M(\alpha)}. \tag{8}$$

will obviously be satisfied.

Taking (8) into account, we have

$$\sup_N \int_0^1 |z_N(t)y(t)|\,dt = \sup_N f(|z_N(t)|\operatorname{sign} y(t)) \leq \|f\| \sup_N \|z_N\|_{M(\alpha)} \leq \|f\|\|x\|_{M(\alpha)} < \infty.$$

By Fatou' s theorem,

$$\left|\int_0^1 x(t)y(t)\,dt\right| \leq \|f\|\|x\|_{M(\alpha)}.$$

In particular, when $x(t) = \theta(t)$, then

$$\int_0^1 x(t)y(t)\,dt = \int_0^1 \theta(t)y(t)\,dt = \int_0^1 \alpha t^{\alpha-1} y^*(t)\,dt = \|y\|_{\Lambda(\alpha)}.$$

Thus, $y(t) \in \Lambda(\alpha)$. Using Lemma 1 and the continuity of the functional $f(x)$, it is easy to show by passage to the limit that formula (6) holds for all $x \in M_0(\alpha)$. The theorem is proved. Thus, $\Lambda(\alpha)$ is conjugate to $M_0(\alpha)$, and $M(\alpha)$ is the second conjugate of $M_0(\alpha)$.

Theorem 2. If $x(t) \in M(\alpha)$, then

$$d(x, M_0(\alpha)) = \lim_{N\to\infty} \|x - x_N\|_{M(\alpha)},$$

where $d(x, M_0(\alpha))$ denotes the distance from the function $x(t)$ to $M_0(\alpha)$, and $x_N(t)$ are the cutoffs of the function $x(t)$.

Proof. It is evidently enough to show

$$d(x, M_0(\alpha)) \geq \lim_{N\to\infty} \|x - x_N\|_{M(\alpha)}.$$

Let $u_N(t) \in M_0(\alpha)$ and

$$d(x, M_0(\alpha)) = \lim_{N\to\infty} \|x - u_N\|_{M(\alpha)}.$$

Since the set of bounded functions is dense in $M_0(\alpha)$, we may assume that $u_N(t)$ are bounded functions. Repeating each function the required number of times, one can arrange that

$$|u_N(t)| \le N \tag{9}$$

for sufficiently large N. If (9) is satisfied, then

$$\|x - u_N\|_{M(\alpha)} \ge \|x - x_N\|_{M(\alpha)}, \qquad d(x, M_0(\alpha)) \ge \lim_{N\to\infty} \|x - x_N\|_{M(\alpha)}.$$

The theorem is proved.

2. In the well-known interpolation theorem of Marcinkiewicz (see ([3])), functions satisfying the condition

$$\sup_{0<\tau<\infty} \tau n_x^{1-\alpha}(\tau) < \infty, \tag{10}$$

are studied, where $n_x(\tau) = mE(|x| > \tau)$.

The functional $\sup_{0<\tau<\infty} \tau n_x^{1-\alpha}(\tau)$, defined on the set of functions for which (10) is satisfied, does not have the properties of a norm (the triangle inequality is not satisfied). However, the following assertion is true.

Lemma 2. The collection of functions satisfying the Marcinkiewicz condition (10) coincides with the space $M(\alpha)$, and moreover

$$\sup_{0<\tau<\infty} \tau n_x^{1-\alpha}(\tau) \le \|x\|_{M(\alpha)} \le \frac{1}{\alpha} \sup_{0<\tau<\infty} \tau n_x^{1-\alpha}(\tau).$$

Proof. If we put $n_x(\tau) = t$, then by the definition of the rearrangement of a function (see ([4]), p. 332), $\tau = x^*(t)$. Therefore it is enough to prove the inequality

$$\sup_{0<t\le 1} x^*(t)t^{1-\alpha} \le \|x\|_{M(\alpha)} \le \frac{1}{\alpha} \sup_{0<t\le 1} x^*(t)t^{1-\alpha}.$$

If

$$\frac{1}{\alpha} \sup_{0<t\leqslant 1} x^*(t)t^{1-\alpha} = C,$$

then $x^*(t) \leqslant C\alpha t^{\alpha-1}$, and

$$\|x\|_{M(\alpha)} = \sup_{0<h\leqslant 1} \frac{\int_0^h x^*(t)\,dt}{h^\alpha} \leqslant \sup_{0<h\leqslant 1} \frac{\int_0^h C\alpha t^{\alpha-1}\,dt}{h^\alpha} = C.$$

On the other hand, since $x^*(t)$ is nonincreasing,

$$\|x\|_{M(\alpha)} = \sup_{0<h\leqslant 1} \frac{\int_0^h x^*(t)\,dt}{h^\alpha} \geqslant \sup_{0<h\leqslant 1} \frac{hx^*(h)}{h^\alpha} = \sup_{0<t\leqslant 1} x^*(t)t^{1-\alpha}.$$

The lemma is proved.

The Marcinkiewicz theorem itself may be formulated as follows. Let $1 \leqslant p_k < q_k < \infty$ $(k = 0, 1)$, $q_0 \neq q_1$. If a linear operator T is a bounded operator from L_{p_k} to $M(1 - 1/q_k)$ $(k = 0, 1)$, then T acts boundedly from L_{p_τ} to L_{q_τ}, where

$$\frac{1}{p_\tau} = \frac{1-\tau}{p_0} + \frac{\tau}{p_1}, \qquad \frac{1}{q_\tau} = \frac{1-\tau}{q_0} + \frac{\tau}{q_1}, \qquad 0 < \tau < 1.$$

Here $M(0)$ is the set of functions $x(t)$ for which $\sup_{0<t\leqslant 1} x^*(t)t < \infty$.

Correspondingly, the interpolation theorems from ([1,5]) can be reformulated.

Theorem 3. *If T is a bounded operator from $M(1-\alpha_k)$ to $M(1-\beta_k)$ $(k = 1, 2)$, $1 > \alpha_k \geqslant \beta_k > 0$, $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$, then T acts boundedly from $M_0(1 - \alpha_\tau)$ to $M_0(1 - \beta_\tau)$, where $\alpha_\tau = (1 - \tau)\alpha_1 + \tau\alpha_2$, $\beta_\tau = (1 - \tau)\beta_1 + \tau\beta_2$.*

Proof. From interpolation theorem 1 in ([1]) it follows that T is a bounded operator from $M(1 - \alpha_\tau)$ to $M(1 - \beta_\tau)$, and

$$\|Tx\|_{M(1-\beta_\tau)} \leqslant C_\tau \|x\|_{M(1-\alpha_\tau)}. \tag{11}$$

It remains to show that if $x \in M_0(1 - \alpha_\tau)$, then also $Tx \in M_0(1 - \beta_\tau)$. For any sufficiently small $\varepsilon > 0$ the inclusion

$$M(1 - \beta_\tau + \varepsilon) \subset L_{\frac{1}{1-\beta_\tau}}$$

holds. By virtue of (4), $M(1 - \beta_\tau + \varepsilon) \subset M_0(1 - \beta_\tau)$. Let $x(t)$ be bounded on $[0, 1]$. If $Tx \in M_0(1 - \beta_\tau)$ for some $0 < \tau < 1$, then, by the preceding, $Tx \in M(1 - \beta_\tau + \varepsilon)$, which contradicts (11). Thus $Tx \in M_0(1 - \beta_\tau)$ for all $0 < \tau < 1$ and all $x \in L_\infty$.

Every function $x(t) \in M_0(1-\alpha_\tau)$ can be approximated by bounded functions u_N (in the norm of $M(1-\alpha_\tau)$). By virtue of (11), the sequence Tu_N is fundamental in $M(1-\beta_\tau)$, and, as was shown, $Tu_N \in M_0(1-\beta_\tau)$. From the closedness of $M_0(1-\beta_\tau)$ in $M(1-\beta_\tau)$ it follows that $Tx \in M_0(1-\beta_\tau)$.

The theorem is proved.

Theorem 4. *If T is a bounded operator from $M_0(1-\alpha_k)$ to $M_0(1-\beta_k)$ $(k=1,2)$, $1 > \alpha_k \geqslant \beta_k > 0$, $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$, then T acts boundedly from $M_0(1-\alpha_\tau)$ to $M_0(1-\beta_\tau)$.*

The proof follows from theorem 1 in ([1]) and theorem 3.

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Note: Figure translations are in progress. See original paper for figures.