G. N. Smirnova

1963

SovietRxiv

Abstract

Full Text

G. N. Smirnova

On Classes of Uniqueness of the Solution of the Cauchy Problem for Parabolic Equations

(Presented by Academician P. S. Novikov on VII 1, 1963)

The paper considers the Cauchy problem for second-order parabolic equations with many independent variables, degenerating at infinity, and also for equations whose coefficients grow at infinity. Classes of functions are found in which the uniqueness theorem holds for the generalized solution and the classical solution of the Cauchy problem. It is shown that these classes cannot be enlarged. The uniqueness class of the generalized solution of the Cauchy problem is defined by certain integral inequalities characterizing the behavior of functions at infinity. The question of uniqueness of solutions of the Cauchy problem was considered in the works ([1–3]) and others (see the survey in ([3])).

Let H be the set of points $(t, x_1, x_2, \dots, x_n)$ of the space E_{n+1} for which $0 < t \leq T$; let Γ be the hyperplane $t = 0$. By $c = \{c_1, c_2, \dots, c_m, \dots\}$ we denote an infinite sequence of positive numbers such that $c_{m+1} - c_m \geq \delta > 0$, $m = 1, 2, \dots$.

Consider in H, for the equation

$$Lu = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}u) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i u) + cu - \frac{\partial u}{\partial t} = f \tag{1}$$

the Cauchy problem with initial condition

$$u\big|_{\Gamma} = \varphi(x). \tag{2}$$

Let

$$a(r_1) \sum_{i=1}^{n} \alpha_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(t,x)\alpha_i\alpha_j \leq A(r_1) \sum_{i=1}^{n} \alpha_i^2, \tag{3}$$

where

$$r_1 = \left(\sum_{i=1}^{n} x_i^2 + 1 \right)^{1/2}, \qquad a(r_1) > 0 \text{ for } r_1 < \infty,$$

$$A(r_1) \in C^1, \qquad c(t,x) \le c_0 < 0;$$

$$\int_1^\infty \frac{dy}{\sqrt{A(y)}} = \infty, \qquad \frac{\sqrt{A(r_1)}}{r_1 \int_1^{r_1} \frac{dy}{\sqrt{A(y)}}} \le M_1, \qquad \frac{\left|\frac{d}{dr_1}(\sqrt{A(r_1)})\right|}{\int_1^{r_1} \frac{dy}{\sqrt{A(y)}}} \le M_2; \quad (4)$$

$$\frac{\sum_{i=1}^{n} |b_i(t,x)|}{\sqrt{A(r_1)} \int_1^{r_1} \frac{dy}{\sqrt{A(y)}}} \le M_3; \tag{5}$$

$$\frac{|f(t,x)|}{\left(\int_1^{r_1} \frac{dy}{\sqrt{A(y)}}\right)^{2-\varepsilon}} \le M_4, \tag{6}$$

where M_i, $i = 1,2,3,4$, c_0 are constants; $0 < \varepsilon < 1$;

$$a_{ij}(t,x),\ b_i(t,x)\ (i,j = 1,2,\dots,n),\ c(t,x),\ f(t,x) \tag{7}$$

are continuous in $\bar{H} \cup \Gamma$ and belong to $C^\alpha(D)$, where D is any bounded domain of H, $0 < \alpha \leqslant 1$.

Obviously, the functions $A(r_1) = r_1^\lambda$ for $-\infty < \lambda \leqslant 2$ satisfy conditions (4).

Definition 1. We shall say that a function $u(t,x)$ belongs to the class $\mathfrak{A}^c$ if $u(t,x) \in L_2(D)$ is summable on the lateral surface S_m^T of the cylinder

$$R_m^T\{r \leqslant c_m,\ 0 \leqslant t \leqslant T\}, \qquad m = 1,2,\dots; \quad r = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2},$$

and, moreover,

$$\int_{S_m^T} |u|\, ds \leqslant \frac{C \exp\left[K\left(\int_1^{\sqrt{c_m^2+1}} \frac{dy}{\sqrt{A(y)}}\right)^2\right]}{A\left(\sqrt{c_m^2+1}\right)}, \tag{8}$$

$m = 1,2,\dots$; C and K are certain positive constants independent of m; ds is the surface-area element of S_m^T.

Definition 2. By a **generalized solution of the Cauchy problem in H for equation (1) with condition (2)** we shall mean a function $u(t,x) \in L_2(D)$, summable on the lateral surface S_m^T of the cylinder R_m^T, such that for every function $\Phi_m(t,x)$ vanishing for $t = \tau$ ($\tau \leqslant T$) and on the lateral surface S_m^τ, the relation

$$\int_0^\tau \iint_{\Omega_m} uL^*\Phi_m \, dx \, dt - \int_0^\tau \iint_{\Omega_m} f\Phi \, dx \, dt + \iint_{\Omega_m} \varphi\Phi_m \, dx - \int_{S_m^\tau} u \left[\sum_{i,j=1}^n a_{ij} \cos(n, x_j) \frac{\partial \Phi_m}{\partial x_i}\right] ds = 0, \tag{9}$$

holds, where Ω_m is the base of the cylinder R_m^T, $dx = dx_1 dx_2 \cdots dx_n$,

$$L^*u = \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_i^n b_i \frac{\partial u}{\partial x_i} + cu + \frac{\partial u}{\partial t}. \tag{10}$$

Obviously, every classical solution of the Cauchy problem for equation (1) with condition (2) is a generalized solution in the sense of Definition 2.

Theorem 1. *Suppose that conditions (3)–(5), (7) are fulfilled; then the generalized solution of the Cauchy problem for equation (1) with condition (2) is unique in the class of functions $\mathfrak{A}^c$.*

Proof. We first show that if $\Phi_m(t,x)$ is a solution of the first boundary-value problem in R_m^τ for the equation

$$L^*\Phi_m = \psi(t,x), \tag{11}$$

vanishing for $t = \tau$ and on S_m^τ, where $\psi(t,x)$ is a finite function in H, and $\tau = \frac{1}{\alpha}$, then

$$\left|\frac{\partial \Phi_{m+1}}{\partial r}\right|_{S_{m+1}^\tau} \leqslant \frac{\exp\left[-M\left(\int_1^{\sqrt{c_{m+1}}^2+1} \frac{dy}{\sqrt{A(y)}}\right)^2 - 1\right]}{c_{m+1} - c_m}.$$

where M is an arbitrary positive constant, and α is a sufficiently large positive number, which we shall determine later. The existence of $\Phi_m(t,x)$ was proved in (3).

Consider in R_m^τ the auxiliary functions

$$w_\pm(t,x) = \exp\left\{-\left[M\left(\int_1^{r_1} \frac{dy}{\sqrt{A(y)}}\right)^2 + 1\right] e^{\alpha t}\right\} \pm \Phi_m(t,x).$$

We shall show that

$$|\Phi_m(t,x)| < \exp\left\{-\left[M\left(\int_1^{r_1}\frac{dy}{\sqrt{A(y)}}\right)^2+1\right]e^{\alpha t}\right\}$$

for any m. Obviously, $w_\pm\big|_{t=\tau} > 0$, $w_\pm\big|_{S_m^\tau} > 0$. It is easy to verify that

$$L^* w_\pm < 0 \quad \text{in } R_m^\tau$$

for some sufficiently large $\alpha > 0$, independent of m. By the maximum principle (3),

$$|\Phi_m(t,x)| < \exp\left\{-\left[M\left(\int_1^{r_1}\frac{dy}{\sqrt{A(y)}}\right)^2+1\right]e^{\alpha t}\right\}, \qquad m = 1, 2, \dots .$$

Consider in $R_{m+1}^\tau \setminus R_m^\tau$ the auxiliary functions

$$v_\pm(t,x) = (c_{m+1} - r + K_1)\gamma \pm \Phi_{m+1}(t,x),$$

where

$$\gamma = \frac{\exp\left\{-\left[M\left(\int_1^{\sqrt{c_m^2+1}}\frac{dy}{\sqrt{A(y)}}\right)^2+1\right]\right\}}{c_{m+1}-c_m};$$

K_1 is a positive constant, which will be determined below.

Obviously,

$$v_\pm\big|_{S_{m+1}^\tau} = K_1\gamma,$$

$$v_\pm\big|_{S_m^\tau} = \exp\left\{-\left[M\left(\int_0^{\sqrt{c_m^2+1}}\frac{dy}{\sqrt{A(y)}}\right)^2+1\right]\right\} + K_1\gamma \pm \Phi_{m+1}\big|_{S_m^\tau} > K_1\gamma,$$

$$v_\pm\big|_{t=\tau} = (c_{m+1} - r)\gamma + K_1\gamma > K_1\gamma,$$

$$L^* v_\pm < 0 \quad \text{in } R_{m+1}^\tau \setminus R_m^\tau,$$

if K_1 and m are sufficiently large. By the maximum principle $v_\pm(t,x)$ attains its minimum on S^τ_{m+1}, and therefore

$$\frac{\partial v_\pm}{\partial r}\bigg|_{S^\tau_{m+1}} \le 0, \qquad \text{i.e.} \qquad \left|\frac{\partial \Phi_{m+1}}{\partial r}\right|_{S^\tau_{m+1}} \le \gamma.$$

Since $\Phi_{m+1} = 0$ on S^τ_{m+1}, the derivatives in the directions tangent to S^τ_{m+1} are equal to zero. Consequently,

$$\left|\frac{\partial \Phi_{m+1}}{\partial x_i}\right| \le \gamma, \qquad i = 1, 2, \dots, n.$$

Let $u_1(t,x)$ and $u_2(t,x)$ be solutions of the Cauchy problem for equation (1) with condition (2), belonging to $\mathfrak{A}^c$. Obviously, $u(t,x) = u_1(t,x) - u_2(t,x)$ satisfies inequality (8) for some positive constants C_0 and K_0. Substituting $u(t,x)$ and $\Phi_m(t,x)$ into the integral identi-

property (9), we obtain that

$$\iint_{R^\tau_m} u\psi\, dx\, dt = \int_{S^\tau_m} u \left[\sum_{i,j=1}^{n} a_{ij} \cos(n, x_j) \frac{\partial \Phi_m}{\partial x_i}\right] ds.$$

Hence

$$\left|\iint_{R^\tau_m} u\psi\, dx\, dt\right| \le \frac{n \exp\left\{-\left[M\left(\int_1^{\sqrt{c_m^2+1}} \frac{dy}{\sqrt{A(y)}}\right)^2 + 1\right]\right\} A\left(\sqrt{c_m^2+1}\right)}{c_{m+1} - c_m} \int_{S^\tau_m} |u|\, ds.$$

Take $M = 2K_0$. Then, in view of the fact that $u(t,x) \in \mathfrak{M}^c$, the right-hand side of the inequality tends to zero as $m \to \infty$. Consequently, since $\psi(t,x)$ is a finite function,

$$\iint_{H\cap[0,\tau]} u\psi\, dx\, dt = 0.$$

It follows that $u(t,x) \equiv 0$ almost everywhere for $0 \le t \le \frac{1}{\alpha}$. Repeating the proof for the points $H \cap [\frac{1}{\alpha} \le t \le \frac{2}{\alpha}]$, then for the points $H \cap [\frac{2}{\alpha} \le t \le \frac{3}{\alpha}]$, etc., we obtain that $u(t,x) \equiv 0$ in H.

Definition 3. We shall say that a function $u(t,x)$ belongs to the class $\mathfrak{M}^\lambda$ if

$$|u(t,x)| \leq C \exp\left[K\left(\int_1^{r_1} \frac{dy}{\sqrt{A(y)}}\right)^{\lambda}\right],$$

where C and K are certain positive constants.

Theorem 2. A solution of the Cauchy problem $u(t,x)$ with condition (2), continuous in $H \cup \Gamma^*$ and satisfying in H the equation

$$L^* u = f(t,x), \tag{12}$$

for whose coefficients conditions (3)–(5) are fulfilled, is unique in the class $\mathfrak{M}^2$.

In the class $\mathfrak{M}^{2+\varepsilon}$, uniqueness of the solution of the Cauchy problem for the equation $L^* u = f$ may fail. This can be shown by means of examples analogous to those constructed in [2].

Theorem 3. A solution of the Cauchy problem in H for equation (12), whose coefficients satisfy conditions (3)–(7), with initial function $\varphi(x) \in \mathfrak{M}^{2-\varepsilon}$, exists in the class $\mathfrak{M}^{2-\varepsilon}$.

In the class $\mathfrak{M}^2$, a solution of the Cauchy problem for equation (12) with initial function $\varphi(x) \in \mathfrak{M}^2$ may fail to exist in H [3].

In the proof of Theorems 2 and 3, auxiliary functions analogous to the functions of paper [3], and the maximum principle, are used.

Theorem 4. A solution of the Cauchy problem for equation (12), whose coefficients satisfy condition (3), is unique in the class of functions having a limit as $r_1 \to \infty$.

I express my deep gratitude to O. A. Oleinik for the advice and comments made in the course of the work.

State Pedagogical Institute
named after V. I. Lenin

Received
24 VI 1963

REFERENCES

1. E. Holmgren, *Ark. f. Math.*, **18** (1924).

2. A. N. Tikhonov, *Matem. sborn.*, **42**, No. 2, 199 (1935).

3. A. M. Il' in, A. S. Kalashnikov, O. A. Oleinik, *Uspekhi Mat. Nauk*, **17**, no. 3, 3 (1962).