ESTIMATION OF THE COMPLETE BEST APPROXIMATION BY PARTIAL BEST APPROXIMATIONS OF FUNCTIONS OF SEVERAL VARIABLES

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Abstract

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ESTIMATION OF THE COMPLETE BEST APPROXIMATION BY PARTIAL BEST APPROXIMATIONS OF FUNCTIONS OF SEVERAL VARIABLES

(Presented by Academician S. N. Bernstein, 8 VI 1963)

Let L_p $(1 \leq p \leq \infty)$ be the space of all functions $f(x_1, \dots, x_k)$ of period 2π in each of the variables x_i $(i = 1, 2, \dots, k)$, whose p-th power of the modulus is integrable on the k-dimensional cube of periods. Denote by $E^{(p)}_{n_1,\dots,n_k}(f)$ the complete best approximation [1] of the function f by trigonometric polynomials of order $\leq n_i$, respectively in the variables x_i $(i = 1, 2, \dots, k)$, and by $E^{(p)}_{n_1,\dots,n_r,\infty}(f)$ the partial best approximation by trigonometric polynomials of order n_i, respectively in the variables x_i $(i = 1, 2, \dots, r;\ r < k)$, with coefficients depending on the variables x_i $(i = r + 1, \dots, k)$ and belonging to the class L_p $(1 \leq p \leq \infty)$.

We shall say that a function $\varphi(x_1, \dots, x_r)$ of the class L_p $(1 \leq p \leq \infty)$ belongs to the set $R_{1,\dots,r}(f)$, if $E^{(p)}_{n_i,\infty}(\varphi) \leq E^{(p)}_{n_i,\infty}(f)$ for $i = 1, 2, \dots, r;\ n_i = 0, 1, 2, \dots$. Put

$$\bar{E}^{(p)}_{n_1,\dots,n_r,\infty}(f) = \inf_{T_{n_1,\dots,n_r}(x_1,\dots,x_k)} \left\| f(x_1, \dots, x_k) - T_{n_1,\dots,n_r}(x_1, \dots, x_k) \right\|_{L_p}, \quad (1)$$

where $T_{n_1,\dots,n_r}(x_1, \dots, x_k)$ is a trigonometric polynomial of degree n_i in the variables x_i $(i = 1, 2, \dots, r)$, with coefficients depending on the variables x_i $(i = r + 1, \dots, k)$ and belonging to the class $R_{r+1,\dots,k}(f)$.

If in this definition $r = k$, then instead of $\bar{E}^{(p)}_{n_1,\dots,n_k,\infty}(f)$ we shall write $\bar{E}^{(p)}_{n_1,\dots,n_k}(f)$.

Theorem 1. If $f \in L_p$ $(1 \leq p \leq \infty)$, then

$$E^{(p)}_{n_1,\dots,n_r,\infty}(f) = \bar{E}^{(p)}_{n_1,\dots,n_r,\infty}(f).$$

Proof. For $r = k$, $R_{r+1,\dots,k}(f)$ will consist only of constants, and therefore

$$E^{(p)}_{n_1,\dots,n_k}(f) = \bar{E}^{(p)}_{n_1,\dots,n_k}(f). \quad (1')$$

Consider the case $r < k$.

By the definition of $E^{(p)}_{n_1,\dots,n_r,\infty}(f)$, for any $\varepsilon > 0$ there exists a trigonometric polynomial $T^{(\varepsilon)}_{n_1,\dots,n_r}(x_1,\dots,x_k)$ of degree n_i in the variables x_i $(i = 1,\dots,r)$, with coefficients depending on x_i $(i = r+1,\dots,k)$ and belonging to the class L_p $(1 \le p \le \infty)$, such that

$$\left\| f(x_1,\dots,x_k) - T^{(\varepsilon)}_{n_1,\dots,n_r}(x_1,\dots,x_k) \right\|_{L_p} \le E^{(p)}_{n_1,\dots,n_r,\infty}(f) + \varepsilon.$$

Moreover, there is such a natural number $N(\varepsilon)$ and such a polynomial $Q_{n_1,\dots,n_k}(x_1,\dots,x_k)$ of degree n_i in x_i $(i = 1,2,\dots,k)$, that

$$\left\| T^{(\varepsilon)}_{n_1,\dots,n_r}(x_1,\dots,x_k) - Q_{n_1,\dots,n_k}(x_1,\dots,x_k) \right\|_{L_p} < \varepsilon \quad \text{for } n_i > N(\varepsilon)$$
$$(i = r+1,\dots,k).$$

Consequently,

$$\overline{E}^{(p)}_{n_1,\dots,n_r,\infty}(f) \le \overline{E}^{(p)}_{n_1,\dots,n_k}(f) = E^{(p)}_{n_1,\dots,n_k}(f) \le \|f - Q_{n_1,\dots,n_k}\|_{L_p} \le$$

$$\le \|f - T^{(\varepsilon)}_{n_1,\dots,n_r}\|_{L_p} + \|T^{(\varepsilon)}_{n_1,\dots,n_r} - Q_{n_1,\dots,n_k}\|_{L_p} \le E^{(p)}_{n_1,\dots,n_r,\infty}(f) + 2\varepsilon.$$

Hence, by the arbitrary smallness of ε,

$$\overline{E}^{(p)}_{n_1,\dots,n_r,\infty}(f) \le E^{(p)}_{n_1,\dots,n_r,\infty}(f). \tag{2}$$

On the other hand, evidently:

$$E^{(p)}_{n_1,\dots,n_r,\infty}(f) \le \overline{E}^{(p)}_{n_1,\dots,n_r,\infty}(f). \tag{3}$$

Therefore, by (2) and (3), we obtain (1). The theorem is proved.

Theorem 2. If $n_i \ge m_i$ $(i = 1,2,\dots,r)$, then

$$E^{(p)}_{n_1,\dots,n_r,n_{r+1},\infty}(T_{m_1,\dots,m_r}) = E^{(p)}_{n_{r+1},\infty}(T_{m_1,\dots,m_r}), \tag{4}$$

where $T_{m_1,\dots,m_r}(x_1,\dots,x_k)$ is a polynomial of degree m_i, respectively, in the variables x_i $(i = 1,2,\dots,r)$, with coefficients depending on the variables x_i $(i = r+1,\dots,k)$ and belonging to the class L_p $(1 \le p \le \infty)$.

Proof. In view of equality (1), it is enough to prove

$$\overline{E}^{(p)}_{n_1,\dots,n_r,n_{r+1},\infty}(T_{m_1,\dots,m_r}) = \overline{E}^{(p)}_{n_{r+1},\infty}(T_{m_1,\dots,m_r}). \tag{5}$$

Obviously, by the definition of $\overline{E}^{(p)}_{n_{r+1},\infty}(f)$, for the function $T_{m_1,\dots,m_r}(x_1,\dots,x_k)$ there exists a polynomial $T^{(\varepsilon)}_{n_{r+1}}(T_{m_1,\dots,m_r};x_1,\dots,x_k)$ of degree n_{r+1} in x_{r+1}, with coefficients belonging to $R_{1,\dots,r,r+2,\dots,k}(T_{m_1,\dots,m_r})$, such that

$$\|T_{m_1,\dots,m_r} - T^{(\varepsilon)}_{n_{r+1}}(T_{m_1,\dots,m_r})\|_{L_p} \le \overline{E}^{(p)}_{n_{r+1},\infty}(T_{m_1,\dots,m_r}) + \varepsilon. \tag{6}$$

It is not difficult to see that the set $R_{1,\dots,r,r+2,\dots,k}(T_{m_1,\dots,m_r})$ will be a subset of all polynomials of degree m_i in the variables x_i $(i=1,2,\dots,r)$ with coefficients depending on the remaining variables x_i $(i=r+1,\dots,k)$ and belonging to the class L_p $(1\le p\le\infty)$. The coefficients of the polynomial $T^{(\varepsilon)}_{n_{r+1}}(T_{m_1,\dots,m_r};x_1,\dots,x_k)$ belong to $R_{1,\dots,r,r+2,\dots,k}(T_{m_1,\dots,m_r})$, and therefore they will be polynomials of degree m_i in x_i $(i=1,2,\dots,r)$.

Consequently, taking (6) into account, we may write

$$\overline{E}^{(p)}_{n_1,\dots,n_r,n_{r+1},\infty}(T_{m_1,\dots,m_r}) \le \|T_{m_1,\dots,m_r} - T^{(\varepsilon)}_{n_{r+1}}(T_{m_1,\dots,m_r})\|_{L_p} \le$$

$$\le \overline{E}^{(p)}_{n_{r+1},\infty}(T_{m_1,\dots,m_r}) + \varepsilon.$$

Hence, by the arbitrary smallness of ε,

$$\overline{E}^{(p)}_{n_1,\dots,n_r,n_{r+1},\infty}(T_{m_1,\dots,m_r}) \le \overline{E}^{(p)}_{n_{r+1},\infty}(T_{m_1,\dots,m_r}). \tag{7}$$

On the other hand,

$$\overline{E}^{(p)}_{n_{r+1},\infty}(T_{m_1,\dots,m_r}) \le \overline{E}^{(p)}_{n_1,\dots,n_r,n_{r+1},\infty}(T_{m_1,\dots,m_r}). \tag{8}$$

From (7) and (8) follows (5), and from (5), in turn, follows (4). The theorem is proved.

Theorem 3. If $f\in L_p$ $(1\le p\le\infty)$, then

$$E^{(p)}_{n_1,\dots,n_k}(f) \le C\sum_{i=1}^{k} E^{(p)}_{n_i,\infty}(f), \tag{9}$$

where C is a constant $\le \dfrac{1}{k}(2^k-1)$.

An inequality of the form (9) for $1 < p < \infty$ was proved by M. F. Timan [1] with an unknown constant depending on p, and for $p = 2,\ k = 2$ by S. N. Bernstein [2] with $C = 1$.

Proof. Let $T^{(\varepsilon)}_{n_{\nu_1},\dots,n_{\nu_{k-1}}}(f; x_1, \dots, x_k)$ be a trigonometric polynomial of degree n_{ν_i}, respectively in the variables x_{ν_i} ($\nu_s = 1, 2, \dots, k$; $s = 1, 2, \dots, k$; $i = 1, 2, \dots, k-1$), with coefficients depending on x_{ν_k} and belonging to L_p ($1 \le p \le \infty$), such that

$$E^{(p)}_{n_{\nu_1},\dots,n_{\nu_{k-1}},\infty}(f) \le \left\| f - T^{(\varepsilon)}_{n_{\nu_1},\dots,n_{\nu_{k-1}}}(f) \right\|_{L_p} \le E^{(p)}_{n_{\nu_1},\dots,n_{\nu_{k-1}},\infty}(f) + \varepsilon.$$

Then, taking equality (4) into account, we can write

$$\begin{aligned} E^{(p)}_{n_1,\dots,n_k}(f) \le E^{(p)}_{n_1,\dots,n_k}\left[f - T_{n_{\nu_1},\dots,n_{\nu_{k-1}}}(f)\right] + E^{(p)}_{n_1,\dots,n_k}\left[T^{(\varepsilon)}_{n_{\nu_1},\dots,n_{\nu_{k-1}}}(f)\right] \le \\ \le E^{(p)}_{n_{\nu_1},\dots,n_{\nu_{k-1}},\infty}(f) + \varepsilon + E^{(p)}_{n_{\nu_k},\infty}\left[T^{(\varepsilon)}_{n_{\nu_1},\dots,n_{\nu_{k-1}}}(f)\right] \le \\ \le E^{(p)}_{n_{\nu_1},\dots,n_{\nu_{k-1}},\infty}(f) + E^{(p)}_{n_{\nu_k},\infty}\left[T^{(\varepsilon)}_{n_{\nu_1},\dots,n_{\nu_{k-1}}}(f) - f\right] + E^{(p)}_{n_{\nu_k},\infty}(f) + \varepsilon \le \\ \le 2E^{(p)}_{n_{\nu_1},\dots,n_{\nu_{k-1}},\infty}(f) + E^{(p)}_{n_{\nu_k},\infty}(f) + 2\varepsilon. \end{aligned}$$

Hence, by the arbitrariness of the small quantity ε, we obtain

$$E^{(p)}_{n_1,\dots,n_k}(f) \le 2E^{(p)}_{n_{\nu_1},\dots,n_{\nu_{k-1}}}(f) + E^{(p)}_{n_{\nu_k},\infty}(f).$$

In exactly the same way one can show that

$$E^{(p)}_{n_{\nu_1},\dots,n_{\nu_{k-1}},\infty}(f) \le 2E^{(p)}_{n_{\nu_1},\dots,n_{\nu_{k-2}},\infty}(f) + E^{(p)}_{n_{\nu_{k-1}},\infty}(f).$$

Continuing this process, it is easy to conclude that

$$E^{(p)}_{n_1,\dots,n_k}(f) \le \sum_{i=1}^{k} 2^{k-1} E^{(p)}_{n_{\nu_i},\infty}(f). \tag{10}$$

ν_i can take k different values for each i; therefore, from (10) we obtain another $k-1$ inequalities of the form (10). Adding all these inequalities and dividing by k, we shall have

$$E^{(p)}_{n_1,\dots,n_k}(f) \le \frac{1}{k}\left(2^{k-1} + 2^{k-2} + \dots + 1\right) \sum_{i=1}^{k} E^{(p)}_{n_i,\infty}(f) = \frac{2^k - 1}{k} \sum_{i=1}^{k} E^{(p)}_{n_i,\infty}(f).$$

The theorem is proved.

The question of whether inequality (9) holds for discontinuous functions when $p = \infty$ was posed by S. N. Bernstein [2]. Theorem 3 gives an affirmative answer to this question.

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References

[1] M. F. Timan, DAN, **112**, No. 1, 24 (1957).
[2] S. N. Bernstein, *Collected Works*, 2, Moscow, 1954, p. 240.

Note: Figure translations are in progress. See original paper for figures.