Yu. I. Manin

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Abstract

Full Text

Yu. I. Manin

PROOF OF AN ANALOGUE OF MORDELL'S CONJECTURE FOR ALGEBRAIC CURVES OVER FUNCTION FIELDS

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1.

The purpose of this note is to announce the proof of the following theorem.

Theorem 1. Let K be a field of characteristic zero, and let $k \subset K$ be its relatively algebraically closed subfield. Suppose that K/k is an extension of finite type. If a curve C of genus $\geqslant 2$, defined over the field K, has infinitely many points rational over K, then there exists a curve C_0, birationally equivalent to C over K and defined over k, such that almost all points of the curve C are images of points of the curve C_0 rational over k.

This result is an analogue for function fields of constants K of the well-known Mordell conjecture ([1]). (For a more detailed discussion see ([2], [3]).)

2.

The main role in the proof of Theorem 1 is played by the technique of additive differential-algebraic functions on abelian varieties, developed partly in ([4]) and in the note ([5]) (the main lemma). The following construction is involved.

Let V be an algebraic variety over the field K; let $\partial_1, \dots, \partial_m$ be a basis of the K-space of derivations of the field K that vanish on k; let U be the K-algebra of differential linear operators generated by the derivations $\partial_1, \dots, \partial_m$. Let Z be the K-space of closed differential forms on V; B the subspace of exact forms. As was shown in ([6]), the algebra U acts naturally on the space Z/B. Let $\omega_1, \dots, \omega_g$ be closed one-dimensional differential forms of the first kind on V. By a **Picard-Fuchs equation** on the variety V we mean any relation of the form

$$\mu : \sum_{i=1}^{g} L_i \overline{\omega}_i = 0, \qquad L_i \in U, \quad \overline{\omega}_i = \omega_i \bmod B.$$

For any transcendence basis $x = (x_1, \dots, x_n)$ of the field $K(V)/K$, the equation μ is determined by its representative

$$\sum_{i=1}^{g} L_{ix}\omega_i = dz, \qquad z \in K(V) \tag{1}$$

(the meaning of the notation L_{ix} is described in ([6])). Suppose that the transcendence basis $x = (x_1, \dots, x_n)$ has the following properties with respect to some pair of simple points $P, Q \in V$, rational over K:

a) $x_i \in O_P \cap O_Q, \quad i = 1, \dots, n;$

b) $\partial_i(x_{jP}) = \partial_i(x_{jQ}) = 0, \quad i = 1, \dots, m;\ j = 1, \dots, n.$

(Here x_P denotes the value of the function x at the point P.) Then put

$$\mu(P, Q) = z_Q - z_P \in K \tag{2}$$

(z is determined from relation (1)). It can be proved that $\mu(P, Q)$ is determined by the Picard–Fuchs equation μ, and does not depend on the arbitrariness of the construc-

···and is uniquely completed to a function defined on all simple pairs of points of the variety V and satisfying identically an equation of the form

$$\mu(P, Q) + \mu(Q, R) = \mu(P, R). \tag{3}$$

Let A be an abelian variety over the field K; let $O \in A_K$ be the zero point of the group law. For any Picard–Fuchs equation μ on A, put $\mu(P) = \mu(O, P)$, $P \in A_K$. From relation (3) and the invariance of differentials of the first kind on A with respect to translations, the following identity is obtained without difficulty:

$$\mu(P) + \mu(Q) = \mu(P + Q). \tag{4}$$

Thus every Picard–Fuchs equation on an abelian variety A determines a certain homomorphism of the group of rational points A_K into the additive group of the field K. Denote by the symbol $A^0_{K/k}$ the intersection of the kernels of all such homomorphisms. The following result plays a fundamental role in the proof of Theorem 1.

Theorem 2. *Let (B, τ) be the K/k-trace of the variety A. The group $A^0_{K/k}$ consists of those points $P \in A_K$ for which there exists an integer $d \neq 0$ such that $dP \in \tau(B_k)$.*

To prove this assertion one has to use topological-analytic means. Standard reductions make it possible to restrict oneself to the case where k is the field of complex numbers and $\dim_k K = 1$. After this one can prove that a fundamental system of solutions of the Picard–Fuchs equations on A is formed by the periods of differentials of the first kind over one-dimensional cycles of A. From the identities $\mu(P) = 0$ for all μ it then follows that the abelian integrals

$$\int_0^p \omega$$

are linearly expressible through the periods ω with complex coefficients. In order to establish that, when the K/k-trace B is zero-dimensional, these coefficients are rational (to this the assertion of Theorem 2 is reduced), we introduce into consideration the monodromy group of the system of Picard–Fuchs equations on A.

This is the only place where topology and analysis are used. All the remaining arguments and constructions are purely algebraic.

3. The application of Theorem 2 to the study of rational points on algebraic curves is based on the fact that the construction of the functions μ has a functorial character. In particular, on a curve C embedded in its Jacobian variety, every function $\mu(P, Q)$ is induced by a similar function on the whole variety. From Theorem 2 and the Mordell–Weil theorem it follows that $\mu(O, P)$, for fixed O and variable point P, runs through a subgroup of finite rank of the field K. It is precisely in this form that Theorem 2 is used to reduce to a contradiction the hypothesis that on the curve C there is an infinite set of points rational over the field K of arbitrarily large height (the height is defined canonically if $\dim_k K = 1$; the general case is reduced to this without difficulty; see Lang ([2,3])). Once such a contradiction has been obtained, Theorem 1 is proved by virtue of Lang' s results ([2]).

Let us illustrate the idea of the proof in one special case. Suppose that on the curve C there are two linearly independent differentials of the first kind $\omega_1 = u_1 dx$, $\omega_2 = u_2 dx$, for which there are Picard–Fuchs equations with representatives of the form

$$\mu_1 : \ \partial_x \omega_1 + \omega_1' = d\omega_1,$$

$$\mu_2 : \ \partial_1 \omega_2 + \omega_2' = d\omega_2.$$

Here ∂ is a nontrivial differentiation of the field K which vanishes on k; ∂_x is its extension to $K(C)$ such that $\partial_x x = 0$. Let

x is a local parameter at the point $O \in C_K$. It can be proved that then

$$\mu_i(P) = u_{iP}\, \partial x_P + w_{iP}, \qquad i = 1, 2, \tag{5}$$

for almost all points $P \in C_K$. Eliminating ∂x_P from equations (5), we obtain the relation

$$\mu_1(P)u_{2P} - \mu_2(P)u_{1P} + (w_1u_2 - w_2u_1)_P = 0. \tag{6}$$

If the function $w_1u_2 - w_2u_1$ were not a linear combination of the functions u_1, u_2 over the field K, then it would follow from formula (6) that almost all points $P \in C_K$ are zeros of functions from a finite-dimensional k-subspace of the field K. Hence it is easy to conclude that the points $P \in C_K$ would have bounded height.

The unfavorable case, when $w_1u_2 - w_2u_1 = c_1u_2 - c_2u_1,\ c_i \in K$, leads to the same conclusion, but in a more roundabout way. Indeed, then

$$(\mu_1(P) + c_1)u_{2P} - (\mu_2(P) + c_2)u_{1P} = 0$$

for almost all $P \in C_K$. Hence, as above, it follows that

$$\mu_i(P) + c_i = 0, \qquad i = 1, 2, \tag{7}$$

for almost all $P \in C_K$.

From equations (5) and (7) it follows that $\partial x_P = y_P$ for some function $y \in K(C)$ and almost all $P \in C_K$. Examining the explicit form of the functions $\mu(P)$ for the remaining Picard–Fuchs equations μ, we can therefore conclude the existence of such functions $x^\mu \in K(C)$ that $\mu(P) = x^\mu_P$ for almost all $P \in C_K$. All Picard–Fuchs equations are consequences of a finite number of them. Hence, from the preceding consideration of zeros of functions from finite-dimensional k-subspaces in the field K, it follows that the points P can have unbounded height only in the case $x^\mu \in K$ for all μ. But in this case the height of these points is bounded by Theorem 2.

If on the curve C it is impossible to find a pair of functions of the form (5), then the investigation becomes technically more complicated, although it uses essentially the same considerations. We cannot dwell on the details.

4. Let us note in conclusion that by the methods of note [5] one can also obtain a strong finiteness theorem for elliptic curves, which contains as a consequence the Siegel–Mahler theorem (in the functional version), but is not covered by it.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.