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Abstract

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ON THE CONTINUATION OF THE SOLUTION OF THE CAUCHY PROBLEM FOR PARABOLIC SYSTEMS

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In the present article the study is continued of the Cauchy problem for nonlinear parabolic systems in the sense of Petrovskii, begun in the dissertation of S. D. Eidelman (^{1}). New uniqueness theorems for the solution of the Cauchy problem for linear systems, obtained by Aronson (^{2}), and certain consequences of interior estimates (^{3}) for solutions defined in a layer (^{4}), in combination with more careful estimates of the sequence of solutions of the Cauchy problem for linear systems whose limit is the solution of a quasilinear (nonlinear) system, have made it possible to obtain very precise theorems on the local solvability of the Cauchy problem for nonlinear parabolic systems and, on their basis, new theorems on continuation of solutions. It turned out that such continuation is always possible in some rather strong norm. Alongside such a norm, a natural norm is introduced, directly connected with the system of equations, and the question of the possibility of continuation in the natural norm is considered. If the solution is continuable in the natural norm, which is estimated a priori, then a nonlocal theorem follows simply from this. An example of application of the last considerations is given at the end of the article.

We shall denote by $\Pi_{(t_0,T]}$ the layer in the $(n+1)$-dimensional space of points (t,x) for which $t \in (t_0,T]$, $x \in E_n$. The layers $\Pi_{[t_0,T]}$ and $\Pi_{(t_0,T)}$ are defined analogously. By $C^{(m,\alpha)}(E_n)$ we shall denote the class of functions defined in E_n and having continuous and bounded derivatives up to order m, satisfying the Hölder condition with exponent α, $C^{(m,0)}(E_n) \equiv C^{(m)}(E_n)$; by $C^{(m,\alpha,\beta)}(\Pi_{[t_0,T]})$ we denote the class of functions $u(t,x)$, defined in $\Pi_{[t_0,T]}$, having continuous and bounded derivatives up to order m, which satisfy the Hölder condition in x and t, respectively with exponents α and β, uniformly in $\Pi_{[t_0,T]}$. Functions $u(t,x)$ defined in $\Pi_{(t_0,T]}$ and $\Pi_{(t_0,T)}$ belong respectively to the class $C^{(m,\alpha,\beta)}(\Pi_{(t_0,T]})$ and $C^{(m,\alpha,\beta)}(\Pi_{(t_0,T)})$, if they satisfy the conditions formulated above in every closed sublayer. Let Q be a domain in the space of points (t,x,y), for which $(t,x) \in \Pi_{[t_0,T)}$, and y belongs to some finite domain of the space E_ν. By $C^{(m,\alpha,\beta,1)}(\overline{Q})$ we denote the class of functions $f(t,x,y)$ which are defined in Q and have there continuous and bounded derivatives up to order m, satisfying the Hölder condition in x,t,y respectively with exponents α,β, and 1, uniformly in Q. For simplicity all results are formulated for one equation.

1. **Local theorems.** We shall consider the Cauchy problem for parabolic

equations of the following three types:

$$\frac{\partial u}{\partial t} = \sum_{|m|=2b} A_m(t, x, D_x^k u) D_x^m u + F_1(t, x, D_x^k u) \equiv$$

$$\equiv P_0(t, x, D_x^k u; D_x) u + F_1(t, x, D_x^k u) \quad (|k| \leq 2b - 1); \tag{1}$$

$$\frac{\partial u}{\partial t} = F_2(t, x, D_x^k u) \quad (|k| \leq 2b); \tag{2}$$

$$\frac{\partial u}{\partial t} = P_0(t, x; D_x) u + F_3(t, x, D_x^k u) \quad (|k| \leq l;\ l \leq 2b - 1); \tag{3}$$

$$u\big|_{t=+t_0} = \varphi(x). \tag{4}$$

By Q here we shall mean the domain described above, where the point y belongs to some cube of the space E_ν. The dimension ν is determined

number of different derivatives entering as arguments in the functions that determine the equations under consideration.

Theorem 1. 1) Let the coefficients of the operator $P_0(t, x, y; D_x)$ and the function $F_1(t, x, y)$ belong to $C^{(1,\alpha,\alpha/2b,1)}(Q)$, $0 < \alpha < 1$, and let the function $\varphi(x) \in C^{(2b+1,\alpha)}(E_n)$. Then problem (1), (4) has a unique solution, defined in $\Pi_{(t_0,t_0+\Delta]}$, $\Delta > 0$, and belonging to $C^{(2b,\alpha,\alpha/2b)}(\Pi_{[t_0,t_0+\Delta]}) \cap C^{(2b+1,\alpha,\alpha/2b)}(\Pi_{(t_0,t_0+\Delta]})$. The length of the interval Δ depends on upper bounds for the moduli of the coefficients, the right-hand side and their derivatives, their Hölder constants, on upper bounds for the moduli of the derivatives of $\varphi(x)$ and the Hölder constant of $D_x^k|\varphi|$, $|k| = 2b+1$; on the numbers a and δ (from the parabolicity condition) and on the dimensions of the domain Q.

2) If $F_2(t, x, y) \in C^{(2,\alpha,\alpha/2b,1)}(Q)$, $\varphi(x) \in C^{(2b+2,\alpha)}(E_n)$, then problem (2), (4) has a unique solution, defined in $\Pi_{(t_0,t_0+\Delta]}$ and belonging to $C^{(2b+1,\alpha,\alpha/2b)}(\Pi_{[t_0,t_0+\Delta]}) \cap C^{(2b+2,\alpha,\alpha/2b)}(\Pi_{(t_0,t_0+\Delta]})$.

3) Let the coefficients $P_0(t, x; D_x)$ belong to $C^{(0,\alpha,\alpha/2b)}(\Pi_{[t_0,T]})$, $F_3(t, x, y) \in C^{(0,\alpha,0,1)}(Q)$, $\varphi(x) \in C^{(l,\alpha)}(E_n)$. Then there exists a unique solution of problem (3), (4) and $(t, x) \in C^{(l,\alpha,\alpha/2b)}(\Pi_{[t_0,t_0+\Delta]}) \cap C^{(2b,\gamma,0]}\Pi_{(t_0,t_0+\Delta]}$ ($\gamma = \alpha$ for derivatives of order $|k| \leqslant 2b - 1$; $\gamma < \alpha$, if $|k| = 2b$).

2. The theorem on continuation of the solution of the Cauchy problem

We shall regard each solution as a trajectory in some space $C^{(p)}(E_n)$. Suppose that the functions determining the equations under consideration are given and satisfy the conditions of the local existence theorem in the domain $Q^{(q)}\{(t,x) \in \Pi_{[t_0,T)},\ y \in \pi_\nu^q\}$, where π_ν^q is the parallelepiped in E_ν defined by the inequalities $|y_i| < M_j$, $j = 1, 2, \dots, \nu$; ν is the number of different derivatives with respect to x up to order q; q is the greatest order of the derivative arguments of the functions determining the equations under consideration; T is the upper bound of all t that occur in our consideration. Denote by Ω_p, $p \geqslant q$, the set of functions $u(x)$ belonging to $C^{(p)}(E_n)$ and such that

$$u_j = \sup_{x \in E_n} |D_x^k u| < M_j, \quad |k| \leqslant p,$$

$j = 1, \dots, \mu$, where μ is the number of different derivatives up to order p; here $M_1, \dots, M_\nu$ are the constants defining π_ν^q, while $M_{\nu+1}, \dots, M_\mu$ are certain fixed constants. Membership of a function $u(x) \in C^{(p)}(E_n)$ in the set Ω_p will be determined with the aid of the norm

$$\|u\|_p = \max_j u_j / M_j$$

(for $u(x) \in \Omega_p$, $\|u\|_p < 1$). By $\widetilde{\Omega}_p$ we shall denote the collection of functions $u(x) \in C^{(p)}(E_n)$ for which $\|u\|_p \leq a$ with $a < 1$.

We shall say that a solution $u(t,x)$, defined for $t \in (t_0, t_1)$, lies in Ω_p if, as a function of x, for every fixed t in (t_0, t_1) it belongs to Ω_p. A solution $u(t,x)$, defined for $t \in (t_0, t_2)$, will be called a continuation in the norm $\|u\|_p$ of the solution $u_1(t,x)$, defined for $t \in (t_0, t_1)$ and lying in Ω_p, if $t_2 > t_1$, $u(t,x) = u_1(t,x)$ when $t \in (t_0, t_1)$, and $u(t,x)$ lies in Ω_p.

The following theorem gives sufficient conditions for the possibility of continuing solutions of parabolic equations.

Theorem 2. Suppose that the solutions of the equations under consideration have the following properties: 1) let $u(t,x)$ be a solution, defined on a finite interval (t_0, τ), belonging to $C^{(m,\alpha,\beta)}(\Pi_{(t_0,\tau)})$ and lying in $\widetilde{\Omega}_p$; then, whatever $\tilde{t} \in [t_0^*, \tau)$ may be chosen, where t_0^* is some arbitrarily fixed number greater than t_0, there is a $\Delta > 0$, independent of $\tilde{t}$, such that the solution $u(t,x)$ can be defined for $t \in (t_0, \tilde{t}+\Delta]$, belongs to $C^{(m,\alpha,\beta)}(\Pi_{(t_0,\tilde{t}+\Delta]})$, and lies in Ω_p (property Π); 2) all derivatives up to order p of the solution $u(t,x)$, defined for $t \in (t_0, \tau)$, satisfy the Hölder condition in t uniformly in $\Pi_{(t_0,\tau)}$ (property Γ). Let $u_1(t,x)$ –

solution, which is defined on a finite half-interval $(t_0, t_1]$, belongs to $C^{(m,\alpha,\beta)}(\Pi_{[t_0,t_1]})$ and lies in $\widetilde{\Omega}_p$; then there exists a unique continuation

$u(t,x)$ in the norm $\|u\|_p$ of the solution $u_1(t,x)$, which is either defined on the half-infinite interval (t_0,∞), or

$$\lim_{t\to T}\|u(t,x)\|_p = 1$$

for some finite T. Here, in the case of equation (1), $m=2b+1$, $p\geq 2b-1$; in the case of equation (2), $m=2b+2$, $p\geq 2b$; and in the case of equation (3), $m=2b$, $p=l$.

3. The possibility of continuing solutions of parabolic equations. Knowing on what the length of the time interval Δ, on which the solution is determined by Theorem 1, depends, and certain consequences (4) of the internal a priori estimates make it possible to prove, by means of the method set forth in (1), the validity of the properties Π and Γ for solutions of the quasilinear equation

$$\partial u/\partial t = P_0(t,x,D_x^k u; D_x)u + F_1(t,x,D_x^{k'}u) \tag{5}$$
$$(|k|\leq l,\ l\leq 2b-2,\ |k'|\leq 2b-1),$$

lying in Ω_{2b-1}. And if one uses the equivalence of solutions of the Cauchy problem for equation (2) and of a special system of equations of type (5), then from the preceding it follows that the conditions Π and Γ are also fulfilled for solutions of (2).

Theorem 3. *If: 1) the coefficients* $P_0(t,x,y;D_x)$ *and* $F_1(t,x,y)$ *belong respectively to* $C^{(1,\alpha,\alpha/2b,1)}(Q^{(l)})$ *and* $C^{(1,\alpha,\alpha/2b,1)}(Q^{(2b-1)})$*; 2)* $F_2(t,x,y)\in C^{(3,\alpha,\alpha/2b,1)}(Q^{(2b)})$*; 3) the coefficients* $P_0(t,x;D_x)$ *belong to* $C^{(0,\alpha,\alpha/2b)}(\Pi_{[t_0,T]})$*, and* $F_3(t,x,y)\in C^{(0,\alpha,0,1)(l)}(Q^{(l)})$*, then the properties* Π *and* Γ*, and consequently the assertions of Theorem 2, hold for solutions of equation (5) belonging to* $C^{(2b+1,\alpha,\alpha/2b)}(\Pi_{[t_0,T]})$ *and lying in* Ω_{2b-1}*, for solutions of equation (2) belonging to* $C^{(2b+3,\alpha,\alpha/2b)}(\Pi_{(t_0,T)})$ *and lying in* Ω_{2b+1}*, and for solutions of equation (3) which belong to* $C^{(2b,\gamma,0)}(\Pi_{[t_0,T)})$ *and lie in* Ω_l*. In other words, solutions with the indicated smoothness can be continued: in the case of equation (5), in the norm* $\|u\|_{2b-1}$*; in the case of equation (2), in the norm* $\|u\|_{2b+1}$*; and in the case of equation (3), in the norm* $\|u\|_l$*.*

4. Natural and unnatural norms. According to Theorem 3, solutions of the Cauchy problem for nonlinear parabolic equations are continuable in some norm $\|u\|_p$, $p\geq q$. We shall call the norm $\|u\|_q$ natural, since only those M_j by which the domain where the equation is specified is determined enter into its definition. The norm $\|u\|_p$, $p>q$, also includes certain constants not connected with the given equation. Theorem 3, in particular, asserts that solutions of equations (3) and (5) are continuable in the natural norm.

In the case when solutions are continued in an unnatural norm, simple arguments using the fact that the definition of this norm includes constants not connected with the domain of specification of the equation, and Theorem 3, give sufficient conditions under which the solution can be continued in the natural norm. These conditions consist in requiring that for the derivatives $D_x^k u(t,x)$, $q+1 \le |k| \le p$, an a priori estimate

$$|D_x^k u(t,x)| \le M(T)$$

hold for any interval (t_0, T), provided that the derivatives up to order q are bounded. Thus, solutions of equation (2) are continued in the natural norm $\|u\|_{2b}$ if derivatives of order $2b+1$ of them are a priori bounded.

5. Nonlocal theorem. In the preceding sections sufficient conditions were given under which solutions of the Cauchy problem are continuable up to the boundary of the set in $C^{(q)}(E_n)$ determined by the domain where the equation is specified. To obtain nonlocal theorems, in addition to this it is necessary to be able to find bounds on the functions defining the equation, under which the following a priori estimate is valid for solutions: for any finite interval (t_0, σ) there is such a finite constant $M(\sigma)$ that for the solution $u(t,x)$ the estimate

$$\|u\|_{C^{(q)}(E_n)} \le M(\sigma).$$

holds. In ⋯

this equation must be prescribed in a domain dictated by the a priori estimates successively obtained for $u(t,x)$ and its derivatives.

As an example illustrating these considerations, let us solve the Cauchy problem for the simplest quasilinear parabolic system

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(t,x) E \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(t,x,u) \frac{\partial u}{\partial x_j} + c(t,x,u), \qquad u = (u_1, \dots, u_N). \tag{6}$$

With the aid of method $(^1)$ it is easy to prove that solutions of the Cauchy problem for system (6) are extendable in the natural norm $\|u\|_0$ (this is established without any restrictions on the growth of b_j and c with respect to u). To prove a nonlocal theorem one needs, in addition, an a priori estimate $|u| \le M(T)$, which is obtained by method $(^1)$ under the assumption that

$$2\left(\sum_{j=1}^{n} b_j(t,x,u) p_j + c(t,x,u), u\right) - 2\sum_{i,j=1}^{n} a_{ij}(p_i, p_j) \le L(|u|^2)\psi(t) \tag{7}$$

for all $u, p_1, \dots, p_n$, $p_j = \partial u / \partial x_j$, and $(t,x) \in \Pi_{[t_0,T]}$, where $L(r)$ is a monotonically nondecreasing function such that

$$\int_a^{\infty} \frac{dr}{L(r)} = \infty, \quad L(\infty) = \infty,$$

and $\psi(t)$ is a positive continuous function.

Theorem 4. *Suppose condition (7) is fulfilled. If*

$a_{ij}(t,x) \in C^{(0,\alpha,\alpha/2)}(\Pi_{[t_0,T]}) \cap C^{(1,\alpha,0)}(\Pi_{[t_0,T]})$, $\quad b_j(t,x,u)$, $c(t,x,u) \in C^{(1,\alpha,0,1)}(Q_T^{(0)})$,

where

$$Q_T^{(0)} = \left\{ (t,x) \in \Pi_{[t_0,T]};\ |u|^2 < f_1^{-1}\left(f(\Phi_0) + \int_{t_0}^{T} \psi(\tau)\, d\tau \right) = M^2(T) \right\}$$

($f_1^{-1}(r)$ is a monotonically increasing function inverse to the function

$$f_1(r) = f(\Phi_0) + \int_{\Phi_0}^{r} \frac{dz}{L(z)}, \qquad \Phi_0 = \sup_{x\in E_n} \sum_{s=1}^{N} \varphi_s^2$$

$\varphi(x) \in C^{(1,\alpha)}(E_n)$), then problem (4), (6) has a unique solution, defined for $t \in (t_0, T)$ and belonging to

$$C^{(3,\gamma,0)}(\Pi_{(t_0,T)}).$$

Let us note that for the system given by N. D. Vvedenskaya (^{5}), condition (7) is fulfilled with the function $L(r^2) = 2r^2[\alpha(r) - 1]$, $\psi(t) \equiv 1$, if $L(r^2)$ is a monotonically nondecreasing function. Thus, for this system, the divergence of the integral

$$\int_a^\infty \frac{dr}{L(r)}$$

is a necessary and sufficient condition for the validity of the nonlocal theorem.

Note added in proof. If, in addition to the method used above, one employs the device of E. Hopf (^{6}), then the following theorem can be established:

Theorem. *If the coefficients of a linear parabolic system and the nonhomogeneity belong to $C^{(0,\alpha,\alpha/2b)}(\Pi_{[t_0,T]})$, then for it the Cauchy problem is correctly solvable in the space*

$$C^{(2b,\alpha,\alpha/2b)}(\Pi_{[t_0,T]}).$$

This makes it possible to establish a proposal strengthening Theorem 1: it suffices to assume that P_0 and F_1 belong to $C^{(0,\alpha,\alpha/2b,1)}(Q)$, $\varphi \in C^{(2b,\alpha)}(E_n)$; then the solution $u(t,x) \in C^{(2b,\alpha,\alpha/2b)}(\Pi_{[t_0,t_0+\Delta]})$, and in item 3 the solution

$$u(t,x) \in C^{(l,\alpha,\alpha/2b)}(\Pi_{[t_0,t_0+\Delta]}) \cap C^{(2b,\alpha,0)}(\Pi_{[t_0,t_0+\Delta]}).$$

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Note: Figure translations are in progress. See original paper for figures.

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