# ON THE LEBESGUE FUNCTION OF ONE LINEAR APPROXIMATION PROCESS

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**Abstract**

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**N. M. Pavlovskii**

# ON THE LEBESGUE FUNCTION OF ONE LINEAR APPROXIMATION PROCESS

*(Presented by Academician A. N. Kolmogorov, 5 X 1962)*

Consider the class $C$ of all continuous functions $f(x,y)$ periodic of period $2\pi$ in $x$ and $y$. Let

$$S_{mn}(f,x,y) = \frac{a_{00}}{4} + \frac{1}{2}\sum_{k=1}^{m}(a_{k0}\cos kx + b_{k0}\sin kx) + \\ + \frac{1}{2}\sum_{l=1}^{n}(a_{0l}\cos ly + c_{0l}\sin ly) + \sum_{k=1}^{m}\sum_{l=1}^{n}(a_{kl}\cos kx\cos ly + b_{kl}\sin kx\cos ly + \\ + c_{kl}\cos kx\sin ly + d_{kl}\sin kx\sin ly), \tag{1}$$

$$\widetilde{S}_{mn}(f,x,y) = \frac{a_{00}^{MN}}{4} + \frac{1}{2}\sum_{k=1}^{m}(a_{k0}^{MN}\cos kx + b_{00}^{MN}\sin kx) + \\ + \frac{1}{2}\sum_{l=1}^{n}(a_{0l}^{MN}\cos ly + c_{0l}^{MN}\sin ly) + \sum_{k=1}^{m}\sum_{l=1}^{n}(a_{kl}^{MN}\cos kx\cos ly + \\ + b_{kl}^{MN}\sin kx\cos ly + c_{kl}^{MN}\cos kx\sin ly + d_{kl}^{MN}\sin kx\sin ly) \tag{2}$$

be, respectively, the double Fourier sum of order $m$ in $x$ and order $n$ in $y$, and the trigonometric polynomial of the same order interpolating the function $f(x,y)$ in the system of equally spaced points $(x_k, y_l)$:

$$x_k = \frac{2k\pi}{M}, \qquad y_l = \frac{2l\pi}{N}, \qquad k = 1,2,\dots, \quad l = 1,2,\dots,$$

$$M = p(2m+1), \qquad N = q(2n+1), \qquad p = 1,2,\dots, \quad q = 1,2,\dots,$$

whose coefficients are computed by the formulas:

$$a_{\mu\nu}^{MN} = \frac{4}{MN} \sum_{k=1}^{M} \sum_{l=1}^{N} f(x_k, y_l) \cos \mu x_k \cos \nu y_l,$$

$$b_{\mu\nu}^{MN} = \frac{4}{MN} \sum_{k=1}^{M} \sum_{l=1}^{N} f(x_k, y_l) \sin \mu x_k \cos \nu y_l,$$

$$c_{\mu\nu}^{MN} = \frac{4}{MN} \sum_{k=1}^{M} \sum_{l=1}^{N} f(x_k, y_l) \cos \mu x_k \sin \nu y_l,$$

$$d_{\mu\nu}^{MN} = \frac{4}{MN} \sum_{k=1}^{M} \sum_{l=1}^{N} f(x_k, y_l) \sin \mu x_k \sin \nu y_l,$$

$$\mu = 0, 1, 2, \dots, m, \qquad \nu = 0, 1, 2, \dots, n.$$

It is easy to see that the polynomial $S_{mn}(f, x, y)$ is the polynomial of best mean-square approximation on the system of points $(x_k, y_l)$ for the function $f(x, y)$, since it minimizes the sum

$$\sum_{k=1}^{M} \sum_{l=1}^{N} \left[f(x_k, y_l) - T_{mn}(x_k, y_l)\right]^2 .$$

Approximations by means of trigonometric polynomials of best mean-square deviation were studied by M. D. Kalashnikov in ([1–3]).

Any system of numbers $\lambda_{kl}^{(m,n)}$, depending on $k, l$ and $m, n$ ($m, n = 1, 2, \dots;\ k = 0, 1, \dots, m+1;\ l = 0, 1, \dots, n+1;\ \lambda_{00}^{(mn)} = 1;\ \lambda_{k,n+1}^{(mn)} = \lambda_{m+1,l}^{(mn)} = 0$), correspondingly defines two linear processes of approximation of the function $f(x, y)$, assigning to each such function a sequence of trigonometric polynomials of the form

$$\begin{aligned} U_{mn}(f, x, y, \lambda) = \frac{a_{00}}{4} + \frac{1}{2} \sum_{k=1}^{m} \lambda_{k0}^{(mn)} (a_{k0} \cos kx + b_{k0} \sin kx) \\ + \frac{1}{2} \sum_{l=1}^{n} (a_{0l} \cos ly + c_{0l} \sin ly) + \sum_{k=1}^{m} \sum_{l=1}^{n} \lambda_{kl}^{(mn)} (a_{kl} \cos kx \cos ly \\ + b_{kl} \sin kx \cos ly + c_{kl} \cos kx \sin ly + d_{kl} \sin kx \sin ly), \end{aligned} \tag{3}$$

$$\widetilde{U}_{mn}(f,x,y,\lambda) = \frac{a_{00}^{MN}}{4} + \frac{1}{2}\sum_{k=1}^{m}\lambda_{k0}^{(mn)}(a_{k0}^{MN}\cos kx + b_{k0}^{MN}\sin kx)$$
$$+\frac{1}{2}\sum_{l=1}^{n}\lambda_{0l}^{(mn)}(a_{0l}^{MN}\cos ly + c_{0l}^{MN}\sin ly) + \sum_{k=1}^{m}\sum_{l=1}^{n}\lambda_{kl}^{(mn)}(a_{kl}^{MN}\cos kx\cos ly$$
$$+ b_{kl}^{MN}\sin kx\cos ly + c_{kl}^{MN}\cos kx\sin ly + d_{kl}^{MN}\sin kx\sin ly). \tag{4}$$

In the present note, under certain natural restrictions on the system of numbers $\lambda_{kl}^{(mn)}$, the asymptotic behavior of the Lebesgue function of the approximation process (4) is established. Using the usual limiting transition, we shall obtain an asymptotic equality for the norm of the approximation process (3). The corresponding result in the case of one variable was obtained earlier by I. M. Ganzburg and A. F. Timan in ([4]).

**Theorem.** If $\Delta_{2k}^2\lambda_{rs}^{(mn)}$, $\Delta_{2l}^2\lambda_{rl}^{(mn)}$, $\Delta_{2kl}^4\lambda_{kl}^{(mn)}$ $(s = 0,1,\dots,n+1;\ r = 0,1,\dots,m+1)$ are of constant sign and $\lambda_{kl}^{(mn)}$ decreases monotonically in $k$ and $l$, and moreover $\Delta_{kl}^2\lambda_{00}^{(mn)} = O(\frac{1}{mn})$, then as $m,n\to\infty$ the following asymptotic equality holds:

$$\sup_{|f(x,y)|\le 1}\left|\widetilde{U}_{mn}(f,x,y,\lambda)\right| = \frac{4}{\pi^2 pq}\,\frac{\left|\cos\left(\frac{2m+1}{2}x - \frac{\pi}{2p}\right)\right|\left|\cos\left(\frac{2n+1}{2}y - \frac{\pi}{2q}\right)\right|}{\sin\frac{\pi}{2p}\,\sin\frac{\pi}{2q}}$$
$$\times\sum_{k=1}^{m}\sum_{l=1}^{n}\frac{\lambda_{kl}^{(mn)}}{(m-k+1)(n-l+1)} + O\left(\sum_{k=1}^{m}\frac{\lambda_{k0}^{(mn)}}{m-k+1}\right)$$
$$+ O\left(\sum_{l=1}^{n}\frac{\lambda_{0l}^{(mn)}}{n-l+1}\right). \tag{5}$$

Let us outline the proof of the theorem. Substituting the values of the coefficients $a_{kl}^{MN}$, $b_{kl}^{MN}$, $c_{kl}^{MN}$, $d_{kl}^{MN}$ into (2), we obtain

$$\widetilde{U}_{mn}(f,x,y,\lambda) = \frac{4}{MN}\sum_{\mu=1}^{M}\sum_{\nu=1}^{N}f(x_\mu,y_\nu)\left[\frac{1}{4} + \frac{1}{2}\sum_{k=1}^{m}\lambda_{k0}^{(mn)}\cos k(x-x_\mu)\right.$$
$$\left.+\frac{1}{2}\sum_{l=1}^{n}\lambda_{0l}^{(mn)}\cos l(y-y_\nu) + \sum_{k=1}^{m}\sum_{l=1}^{n}\lambda_{kl}^{(mn)}\cos k(x-x_\mu)\cos l(y-y_\nu)\right]. \tag{6}$$

In what follows, for brevity, we put $\lambda_{kl}^{(mn)} = \lambda_{kl}$.

After applying Abel's transformation to the expression standing in square brackets in relation (6), we obtain

$$\widetilde{U}_{mn}(f,x,y,\lambda)=\frac{4}{MN}\sum_{\mu=1}^{M}\sum_{\nu=1}^{N}f(x_\mu,y_\nu)\sum_{k=0}^{m}\sum_{l=0}^{n}\Delta_{kl}^2\lambda_{kl}D_k(x-x_\mu)D_l(y-y_\nu)$$
$$=\frac{4}{MN}\sum_{\mu=1}^{M}\sum_{\nu=1}^{N}f(x_\mu,y_\nu)\left[\sum_{k=0}^{m}\sum_{l=0}^{n}\Delta_{kl}^2\lambda_{m-k,n-l}D_{m-k}(x-x_\mu)D_{n-l}(y-y_\nu)\right], \quad (7)$$

where $D_m(u)$ is the Dirichlet kernel of order $m$,

$$\Delta_{kl}^2\lambda_{kl}=\lambda_{kl}-\lambda_{k+1,l}-\lambda_{k,l+1}+\lambda_{k+1,l+1}.$$

Introduce the notation

$$\sigma_{mn}(f,x,y,k,l)=\frac{1}{MN(k+1)(l+1)}\sum_{\mu=1}^{M}\sum_{\nu=1}^{N}f(x_\mu,y_\nu)\times$$
$$\times\frac{\sin\frac{2m+1-k}{2}(x-x_\mu)\,\sin\frac{k+1}{2}(x-x_\mu)\,\sin\frac{2n+1-l}{2}(y-y_\nu)\,\sin\frac{l+1}{2}(y-y_\nu)}{\sin^2\frac{1}{2}(x-x_\mu)\,\sin^2\frac{1}{2}(y-y_\nu)}.$$

Applying Abel' s transformation to the sum standing in square brackets in relation (6), we obtain

$$\widetilde{U}_{mn}(f,x,y,\lambda)=\sum_{k=0}^{m-n}\sum_{l=0}^{n-1}(k+1)(l+1)\Delta_{kkll}^4\lambda_{m-k-1,n-l-1}\sigma_{mn}(f,x,y,k,l)+$$
$$+(m+1)(n+1)\Delta_{kl}^2\lambda_{00}\sigma_{mn}(f,x,y,m,n)-$$
$$-\sum_{k=0}^{m-1}(k+1)(n+1)\Delta_{kkl}^3\lambda_{m-k-1,0}\sigma_{mn}(f,x,y,k,n)-$$
$$-\sum_{l=0}^{n-1}(m+1)(l+1)\Delta_{kll}^3\lambda_{0,n-l-1}\sigma_{mn}(f,x,y,m,l).$$

Using the identities

$$(n+1)\Delta_{kkl}^3\lambda_{m-k-1,0}=\Delta_{kk}^2\lambda_{m-k-1,0}+\sum_{l=0}^{n-1}(l+1)\Delta_{kkll}^4\lambda_{m-k-1,n-l-1},$$

$$(m+1)\Delta_{kll}^3\lambda_{0,n-l-1}=\Delta_{ll}^2\lambda_{0,n-l-1}+\sum_{k=0}^{m-1}(k+1)\Delta_{kkll}^4\lambda_{m-k-1,n-l-1},$$

$$(m+1)\Delta_{kl}^2\lambda_{00}=\Delta_l\lambda_{00}+\sum_{k=0}^{m-1}(k+1)\Delta_{kkl}^3\lambda_{m-k-1,0},$$

$$(n+1)\Delta_l\lambda_{00}=\lambda_{00}+\sum_{l=0}^{n-1}(l+1)\Delta_{ll}^2\lambda_{0,n-l-1},$$

we give $\widetilde{U}_{mn}(f,x,y,\lambda)$ the following form:

$$\widetilde{U}_{mn}(f,x,y,\lambda)=\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}(k+1)(l+1)\Delta^4_{kkll}\lambda_{m-k-1,n-l-1}\times$$
$$\times\left[\sigma_{mn}(f,x,y,k,l)-\sigma_{mn}(f,x,y,k,n)-\sigma_{mn}(f,x,y,m,l)+\sigma_{mn}(f,x,y,m,n)\right]$$
$$+\sum_{k=0}^{m-1}(k+1)^2\Delta^2_{kk}\lambda_{m-k-1,0}[\sigma_{mn}(f,x,y,m,n)-\sigma_{mn}(f,x,y,k,n)]$$
$$+\sigma_{mn}(f,x,y,m,n)+\sum_{l=0}^{n-1}(l+1)\Delta^2_{ll}\lambda_{0,n-l-1}[\sigma_{mn}(f,x,y,m,n)-$$
$$-\sigma_{mn}(f,x,y,m,l)].$$

Taking into account the equality

$$|\sigma_{mn}(1,x,y,k,l)|=\frac{4}{\pi^2pq}\ln\frac{m}{k+1}\ln\frac{n}{l+1}\times$$
$$\times\frac{\left|\cos\left(\frac{2n+1}{2}x-\frac{\pi}{2p}\right)\right|\left|\cos\left(\frac{2n+1}{2}y-\frac{\pi}{2q}\right)\right|}{\sin\frac{\pi}{2p}\,\sin\frac{\pi}{2q}}+O\Big(\ln\frac{m}{k+1}\Big)+O\Big(\ln\frac{n}{l+1}\Big),$$

and also some relations from (5), we obtain

$$V_{mn}(x,y,\lambda)=\sup_{|f|\leqslant 1}|U_{mn}(f,x,y,\lambda)|\leqslant$$

$$\leqslant\frac{4}{\pi^2pq}\frac{\left|\cos\left(\frac{2m+1}{2}x-\frac{\pi}{2p}\right)\right|\left|\cos\left(\frac{2n+1}{2}y-\frac{\pi}{2q}\right)\right|}{\sin\frac{\pi}{2p}\sin\frac{\pi}{2q}}\sum_{k=1}^{m}\sum_{l=1}^{n}\frac{\lambda_{kl}}{(m-k+1)(n-l+1)}+$$

$$+O\left(\sum_{k=1}^{m}\frac{\lambda_{k0}}{m-k+1}\right)+O\left(\sum_{l=1}^{n}\frac{\lambda_{0l}}{n-l+1}\right).$$

It is not hard to verify that the last inequality becomes an equality for the function $\varphi_{xy}(t,z)$

$$\varphi_{xy}(t,z)=\operatorname{sign}D_m(x-t)D_n(y-z),$$

i.e., the estimate obtained in (5) is asymptotically sharp.

Let us note that, as $p,q\to\infty$, from relation (5) we obtain the asymptotic expression for the Lebesgue function corresponding to the linear approximation process (3):

$$V_{mn}(x,y,\lambda)=\frac{16}{\pi^4}\sum_{k=1}^{m}\sum_{l=1}^{n}\frac{\lambda_{kl}^{(mn)}}{(m-k+1)(n-l+1)}+$$

$$+O\left(\sum_{k=1}^{m}\frac{\lambda_{k0}^{(mn)}}{m-k+1}\right)+O\left(\sum_{l=1}^{n}\frac{\lambda_{0l}^{(mn)}}{n-l+1}\right).$$

Putting $p=q=1$ in relation (5), we obtain an asymptotic estimate of the Lebesgue function for the linear process

$$\frac{a_{00}^{(mn)}}{4}+\frac{1}{2}\sum_{k=1}^{m}\lambda_{k0}^{(mn)}\left(a_{k0}^{(mn)}\cos kx+b_{k0}^{(mn)}\sin kx\right)+$$

$$+\frac{1}{2}\sum_{l=1}^{n}\lambda_{0l}^{(mn)}\left(a_{0l}^{(mn)}\cos ly+c_{0l}^{(mn)}\sin ly\right)+$$

$$+\sum_{k=1}^{m}\sum_{l=1}^{n}\lambda_{kl}^{(mn)}\left(a_{kl}^{(mn)}\cos kx\cos ly+b_{kl}^{(mn)}\sin kx\cos ly+\right.$$

$$\left.+c_{kl}^{(mn)}\cos kx\sin ly+d_{kl}^{(mn)}\sin kx\sin ly\right),$$

where $a_{kl}^{(mn)},b_{kl}^{(mn)},c_{kl}^{(mn)},d_{kl}^{(mn)}$ are the coefficients of the ordinary double interpolating trigonometric polynomial with equidistant nodes $(x_k,y_l)$

$$x_k=\frac{2k\pi}{2m+1},\qquad y_l=\frac{2l\pi}{2n+1},\qquad k=0,\pm1,\pm2,\dots,\pm m,$$

$$l=0,\pm1,\pm2,\dots,\pm n,$$

$$V_{mn}(x,y,\lambda)=\frac{4}{\pi^2}\left|\sin\left(m+\frac{1}{2}\right)x\right|\left|\sin\left(n+\frac{1}{2}\right)y\right|\times$$

$$\times\sum_{k=1}^{m}\sum_{l=1}^{n}\frac{\lambda_{kl}^{(mn)}}{(m-k+1)(n-l+1)}+O\left(\sum_{k=1}^{m}\frac{\lambda_{k0}^{(mn)}}{m-k+1}\right)+O\left(\sum_{l=1}^{n}\frac{\lambda_{0l}^{(mn)}}{n-l+1}\right).$$

Dnepropetrovsk Civil Engineering Institute

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*