L. I. Kamynin

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Abstract

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L. I. Kamynin

ON THE LINEAR VERIGIN PROBLEM

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In a number of questions of physics it is of interest to investigate problems for a parabolic equation with a free boundary, i.e., problems in which it is required to find not only the solution of the equation, but also the boundary of the domain (or part of it) in which this solution is considered. Such are, for example, the well-known Stefan problem on freezing (see ([1])) or the Verigin problem ([2]–[5]), in which one seeks the solution of a mixed problem for a parabolic equation with variable coefficients, discontinuous upon crossing a moving and a priori unknown line separating two phases. In this note we shall consider the Verigin problem, which arises in the practice of hydraulic engineering in the study of the process of forcing fluids into a porous medium. The formulation of the Verigin problem is somewhat reminiscent of the Stefan problem, but differs essentially from it (see in more detail ([5])). The Verigin problem for a one-dimensional parabolic equation with constant coefficients was considered in ([2], [3]). In our papers ([4], [5]) the Verigin problem was studied for a homogeneous parabolic equation with variable coefficients; however, the equation was assumed to be of a very special form. In the present note, developing the methods of ([4], [5]), we consider the Verigin problem for a general linear one-dimensional parabolic equation with boundary conditions of the first, second, and third kinds and with more general conjugation conditions on the line of discontinuity, in the following formulation.

It is required to find three functions $u_1(x,t)$, $u_2(x,t)$, and $h(t)$, satisfying the parabolic equations

$$a_i(x,t)\frac{\partial^2 u_i}{\partial x^2} + b_i(x,t)\frac{\partial u_i}{\partial x} + c_i(x,t)u_i - \frac{\partial u_i}{\partial t} = f_i(x,t), \tag{1}$$

$$0 < t < T, \quad X_1(t) < x < h(t) \text{ for } i = 1, \qquad h(t) < x < X_2(t) \text{ for } i = 2,$$

with the initial conditions

$$u_i(x,0) = \psi_i(x), \tag{2}$$

$$0 = X_1(0) \le x \le c = h(0) \text{ for } i = 1, \qquad c \le x \le l = X_2(0) \text{ for } i = 2,$$

the boundary conditions of the third (second) kind

$$\frac{\partial u_i(X_i(t),t)}{\partial x} + (-1)^i d_i(t) u_i(X_i(t),t) = \varphi_i(t), \qquad 0 \le t \le T, \quad i = 1,2 \tag{3a}$$

or the boundary conditions of the first kind

$$u_i(X_i(t),t) = \mu_i(t), \qquad 0 \le t \le T, \quad i = 1,2, \tag{3b}$$

the conjugation conditions on the unknown line of phase separation $x = h(t)$

$$u_1(h(t),t) - p_1(h(t),t) u_2(h(t),t) = p_2(h(t),t), \qquad 0 \le t \le T, \tag{4}$$

$$\sum_{i=1}^{2} \left[(-1)^{i+1} \lambda_i(h(t),t) \frac{\partial u_i(h(t),t)}{\partial x} - q_i(h(t),t) u_i(h(t),t) \right] = q_3(h(t),t);$$

$$\frac{dh(t)}{dt} = r_1(t) + r_2(t) h(t) - \gamma(h(t),t) \frac{\partial u_1(h(t),t)}{\partial x}, \tag{5}$$

where

$$\gamma(x,t) = [B(t)\lambda_2(x,t)]^{-1} [\lambda_1(x,t) a_2(x,t) - \lambda_2(x,t) a_1(x,t)],$$

and the compatibility conditions

$$\psi_1(c) = \psi_2(c), \qquad p_1(c,0) = 1, \qquad p_2(c,0) = 0,$$

$$\psi_i'(X_i(0)) + (-1)^i d_i(0) \psi_i(X_i(0)) = \varphi_i(0) \quad \text{or} \quad \mu_i(0) = \psi_i(X_i(0)), \qquad i = 1,2, \tag{6}$$

$$\sum_{i=1}^{2} \left[(-1)^{i+1} \lambda_i(c,0) \psi_i'(c) - q_i(c,0) \psi_i(c) \right] = q_3(c,0).$$

The note proves the existence of a solution of Verigin' s problem (1)–(6) under sufficiently broad smoothness conditions on the functions entering into (1)–(6). The study uses the apparatus of heat potentials, the results of Gevrey ([11]) for linear parabolic equations, and the author' s results ([6]–[8]) in the theory of boundary-value problems for a parabolic equation with discontinuous coefficients.

Let the following conditions I–X be fulfilled. I. The functions $X_i(t)$ have derivatives $X_i'(t)$ satisfying the Hölder condition, and $0 < c < l$ and $\min |X_2(t) - X_1(t)| > 0$. In what follows we shall assume $X_1(t) \equiv 0,\ X_2(t) \equiv l$, i.e. we shall consider Verigin' s problem (1)–(6) for the rectangle $D[T] = \{(x,t);\ 0 \leq x \leq l,\ 0 \leq t \leq T\}$, since, if condition I is fulfilled, one can pass to it by means of the change (cf. [11]) $x_1 = (x - X_1(t))[X_2(t) - X_1(t)]^{-1}$. II. Equation (1) is of parabolic type in $D[T]$. III. The coefficients $a_i(x,t), b_i(x,t), c_i(x,t)$ $(i = 1,2)$ are continuous in $D[T]$ with respect to x and t, together with $\partial a_i/\partial x,\ \partial a_i/\partial t,\ \partial b_i/\partial x,\ \partial c_i/\partial x$, and $\partial a_i/\partial x,\ \partial a_i/\partial t,\ b_i$, and c_i satisfy in $D[T]$, with respect to x and t, the Hölder condition. IV. The right-hand side $f_i(x,t)$ in (1) is continuous with respect to x and t in $D[T]$, together with $\partial f_i/\partial x$. V. The initial function $\psi_i(x)$ $(i = 1,2)$ has a derivative ψ_i' satisfying on the intervals $[0,c]$ $(i = 1)$ and $[c,l]$ $(i = 2)$ the Hölder condition. VI. The functions $\varphi_i(t)$ and $d_i(t) \geq 0$ $(i = 1,2)$ satisfy on $[0,T]$ the Hölder condition. The functions $\mu_i(t)$ $(i = 1,2)$ have derivatives μ_i' continuous on $[0,T]$. The functions $q_j(x,t)$ $(j = 1,2,3)$ and $\lambda_i(x,t)$ $(i = 1,2)$, where $\lambda_2(x,t) \geq \lambda_0 > 0$ (λ_0 is constant), satisfy in x and t in $D[T]$ the Hölder condition. The functions $p_i(x,t)$ $(i = 1,2)$ satisfy in x and t in $D[T]$ the Hölder condition with exponent $> 1/2$, and $p_1(x,t) \geq p_0 > 0$ (p_0 is constant). VII. The compatibility conditions (6) are fulfilled. VIII. $B(t)$ is a positive monotonically nondecreasing function for which $[B^{-1}(t)]'$ satisfies on $[0,T]$ the Hölder condition. IX. The functions $r_i(t)$ $(i = 1,2;\ r_1(t) \geq 0)$ satisfy on $[0,T]$ the Hölder condition. X. $\lambda_1(x,t)a_2(x,t) - \lambda_2(x,t)a_1(x,t) \neq 0,\ (x,t) \in D(T)$.

If there exists a solution of Verigin' s problem (1)–(6), where $h(t)$ is a monotonically nondecreasing differentiable function, then (cf. [5]) the identity holds

$$h(t) = [B(t)]^{-1} F(t; u_1, u_2, h). \tag{7}$$

Let $D[t,1] = \{(x,\tau);\ 0 \leq x \leq h(\tau),\ 0 \leq \tau \leq t\}$, $D[t,2] = \{(x,\tau);\ h(\tau) \leq x \leq l,\ 0 \leq \tau \leq t\}$. Let $h(t)$ be a known monotonically nondecreasing function satisfying on $[0,T]$ the Lipschitz condition, and let $u_i(x,t;h)$ be a solution of the auxiliary problem (1)–(4), (6) for a parabolic equation with discontinuous coefficients with line of discontinuity $x = h(t)$, where $h(t)$ is the function specified above. Such a solution $u_i(x,t;h)$, according to our works [6,8], exists if conditions I–VII are fulfilled, and $u_i(x,t;h)$, together with $\partial u_i(x,t;h)/\partial x$, satisfies the Hölder condition in x and t in the closed domain $D[T,i]$ $(i = 1,2)$ (cf. [5]). Define the mapping $g = Sh$ by means of the relation

$$g(t) = B^{-1}(t) F(t; u_1, u_2, h) \equiv Sh(t), \tag{8}$$

where $F(t; u_1, u_2, h)$ in (8) is taken from (7), with the solutions $u_i(x,t;h)$ of the auxiliary problem (1)–(4), (6) substituted in place of $u_i(x,t)$, and $h(t)$ taken from the equation of the line of discontinuity. We note that $g(0) = c$ and $g(t)$ has the derivative

$$g'(t) = [B^{-1}(t)]' \left[F(t; u_1, u_2, h) - B(t)h(t)\right] -$$

$$-\gamma(h(t), t)\frac{\partial u_1(h(t), t; h)}{\partial x} + r_1(t) + r_2(t)h(t). \tag{9}$$

If

$$c_2(x,t) \le c_1(x,t), \qquad c_2(0,0) = c_1(0,0),\ c_2(l,0) = c_1(l,0), \tag{10}$$

then for the solution $z_1(x,t)$ of the first boundary-value problem

$$a_2(x,t)\frac{\partial^2 z_1}{\partial x^2} + b_2(x,t)\frac{\partial z_1}{\partial x} + [c_2(x,t) - c_1(x,t)]z_1 - \frac{\partial z_1}{\partial t} = 0, \quad (x,t) \in D[T],$$

$$z_1(x,0) \equiv 1, \quad 0 \le x \le l; \qquad z_1(0,t) \equiv z_1(l,t) \equiv 1, \quad 0 \le t \le T,$$

the $(2+\alpha)$-estimate of Friedman–Browder (12) holds, and by virtue of Nirenberg's strict maximum principle (13), $0 < \inf z_1(x,t) \le z_1(x,t) \le 1$. Further, if

$$f_2(x,t) \le z_1(x,t)f_1(x,t), \qquad (x,t) \in D[T], \tag{11}$$

then for the solution $z_2(x,t)$ of the first boundary-value problem

$$a_2(x,t)\frac{\partial^2 z_2}{\partial x^2} + z_1^{-1}(x,t)\left[2a_2(x,t)\frac{\partial z_1}{\partial x} + b_2(x,t)\right]\frac{\partial z_2}{\partial x} + c_1(x,t)z_2 - \frac{\partial z_2}{\partial t}$$

$$= z_1^{-1}(x,t)f_2(x,t) - f_1(x,t), \qquad (x,t) \in D[T];$$

$$z_2(x,0) \equiv 0, \quad 0 \le x \le l; \qquad z_2(0,t) \equiv z_2(l,t) \equiv 0, \quad 0 \le t \le T,$$

the estimates

$$0 \le z_2(x,t) \le M_1, \qquad \left|\frac{\partial z_2(x,t)}{\partial x}\right| \le M_1, \qquad (x,t) \in D[T].$$

Lemma 1. Suppose that, in addition to I–VII, (10), (11) are satisfied and

$$b_1(x,t) \geq z_1^{-1}(x,t)\left[2a_2(x,t)\frac{\partial z_1(x,t)}{\partial x} + b_2(x,t)\right], \qquad (x,t)\in D[T]; \qquad (12)$$

then for the solution of the auxiliary problem (1)–(4), (6) the estimates

$$|u_i(x,t;h)| \leq M_2, \qquad \left|\frac{\partial u_i(x,t;h)}{\partial x}\right| \leq M_3, \qquad (x,t)\in D[T,i], \qquad (13)$$

hold, where the constants M_2 and M_3 do not depend on $h(t)$. If, moreover,

$$f_1(x,t) - f_2(x,t) \geq M_2\left[c_1(x,t) - c_2(x,t)\right], \qquad \frac{\partial f_i(x,t)}{\partial x} < 0,$$

$$-\sup\frac{\partial f_i(x,t)}{\partial x} \geq M_2 \sup\frac{\partial c_i(x,t)}{\partial x}, \qquad \frac{\partial f_i(x,t)}{\partial x} \leq M_2\frac{\partial c_i(x,t)}{\partial x}, \qquad (14)$$

$$\psi_i'(x) \leq 0, \qquad \varphi_i(t) < 0, \qquad |\varphi_i(t)| > M_2 d_i(t), \qquad 0 \leq t \leq T \quad \text{for (3a)},$$

$$\mu_i(t)\exp(-\theta t) > (\theta + c_2)^{-1}\sup|f_j(x,\tau)\exp(-\theta\tau)|, \qquad 0 \leq \tau \leq t,$$

$$\frac{d}{dt}\left[\mu_i(t)\exp(-\theta t)\right](-1)^{i+1} \geq 0, \qquad \frac{\partial b_i(x,t)}{\partial x} + c_i(x,t) \leq -c_2,$$

$$\theta + c_2 > 0 \qquad \text{for (3b)},$$

then, along with (13), one has

$$\frac{\partial u_i(x,t;h)}{\partial x} \leq 0, \qquad (x,t)\in D[T,i].$$

The proof of Lemma 1 is carried out according to the scheme of Lemma 1 of [5], with the use of [9,10]; moreover, for the proof of the second of the estimates (13) in equation (1) one first makes the substitution $v_1(x,t;h) \equiv u_1(x,t;h)$; $v_2(x,t;h) \equiv z_2(x,t) + z_1^{-1}(x,t)u_2(x,t;h)$, as a result of which, in the new equations, $c_i(x,t) \equiv c_1(x,t)$, $f_i(x,t) \equiv f_1(x,t)$.

Lemma 2 (see Lemma 1 and (9)). If conditions I–VIII, (10)–(12) are satisfied and

$$r_1(t) + r_2(t)h'(t) > M_4, \qquad 0 \le t \le T, \tag{15}$$

where the constant M_4 depends on the data of problem (1)–(6), then under the mapping-

or (8)

$$0 \leqslant g'(t) \leqslant M_5, \qquad 0 \leqslant t \leqslant T, \tag{16}$$

where the constant M_5 does not depend on $h(t)$. If

$$B'(t) \equiv 0, \tag{17}$$

then in (15)

$$M_4 = M_3 \sup |\gamma(x,t)|, \tag{18}$$

and in (16)

$$M_5 = M_4 + \sup[r_1(t) + l|r_2(t)|]. \tag{19}$$

Finally, if conditions I–VIII, (10)–(12), (14), (17) are satisfied and

$$r_i(t) \geqslant 0, \quad 0 \leqslant t \leqslant T; \qquad \gamma(x,t) \geqslant 0, \quad (x,t) \in D[T], \tag{20}$$

then in (16)

$$M_5 = \sup(r_1(t) + l r_2(t)) + M_3 \sup \gamma(x,t). \tag{21}$$

Denote by $H(M)$ the set whose elements are monotonically nondecreasing functions $h(t)$ $(0 \leqslant t \leqslant T)$, satisfying the Lipschitz condition with constant $M = M_5$ (see, respectively, (16), (19), (21)) and the inequality

$$0 < h(0) = c \leqslant h(t) \leqslant \min(l-d, MT+c), \quad 0 \leqslant t \leqslant T, \quad \text{where } l-c > d > 0.$$

Introduce on $H(M)$ the Lipschitz norm and consider on $H(M)$ the operator (8). From our work [7] it follows that S is continuous on $H(M)$ in the Lipschitz norm; moreover, according to the scheme of work [5] (see [5], Lemmas 5–12), compactness of $S(H(M))$ is proved. From (9), with the aid of Lemmas 1 and 2, follows the existence of at least one fixed point of the mapping $g = Sh$ of the set $H(M)$ into itself.

Theorem 1. Suppose that conditions I–X, (10)–(12) are satisfied and

$$\begin{aligned} &\min(r_1(t) + lr_2(t)) \geqslant M_4, \quad \text{if } \min r_2(t) < 0; \\ &\min(r_1(t) + cr_2(t)) \geqslant M_4, \quad \text{otherwise,} \end{aligned} \tag{22}$$

where M_4 is taken from (15). Then there exists at least one solution $u_1(x,t)$, $u_2(x,t)$, $h(t)$ of the Verigin problem (1)–(6), for which:

A. The functions $u_i(x,t)$, together with $\partial u_i(x,t)/\partial x$, satisfy the Hölder condition in $x, t \in D[T, i]$.

B. The monotonically nondecreasing function $h(t)$ is differentiable, and $h'(t)$ satisfies the Hölder condition.

Theorem 2. If conditions I–X, (10)–(12), (17), (22) are satisfied, where M_4 is taken from (18), then there exists at least one solution of the Verigin problem (1)–(6) having properties A and B of Theorem 1.

Theorem 3. Suppose that conditions I–X, (10)–(12), (17), (20), (14) are satisfied; then there exists at least one solution of the Verigin problem (1)–(6) having properties A and B of Theorem 1.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.