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Abstract

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MATHEMATICS

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SOME EXTREMAL PROBLEMS IN THE CLASS OF POLYNOMIALS AND RATIONAL FUNCTIONS

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Let

$$M(f, r) = \max_{|z|=r} |f(z)|.$$

It is known ([1], p. 167) that for any polynomial of degree n the inequality

$$M(P_n, R) \leq R^n M(P_n, 1), \tag{1}$$

holds, where $R \geq 1$, and for $r \leq 1$

$$M(P_n, r) \geq r^n M(P_n, 1). \tag{2}$$

Inequalities (1) and (2) are sharp in the class of polynomials of degree n.

Denote by $K[n, d]$ the class of polynomials of degree n having no zeros in the disk $|z| < d$. For this class of polynomials, inequalities (1) and (2) have been sharpened by various authors.

In the paper ([2]) it was proved that if $P_n(z) \in K[n, 1]$ and $R \geq 1$, then

$$M(P_n, R) \leq \frac{1 + R^n}{2} M(P_n, 1). \tag{3}$$

This inequality was generalized in the metric of the space $\mathcal{L}_p$ by I. I. Ibragimov and R. G. Mamedov ([3]).

We ([4]) sharpened this result for the class $K[n, \sqrt{R}]$, and it was also proved that if $P(z) \in K[n, 1]$ and $r \leq 1$, then

$$|P_n(re^{i\alpha})| \geq \left(\frac{1+r}{2}\right)^n |P_n(e^{i\alpha})|.$$

For a rational function of the form

$$R(z) = \frac{P_n(z)}{Q_m(z)} = \frac{a_1 z^n + \cdots + a_n}{b_1 z^m + \cdots + b_m}$$

it was proved in the same paper ([4]) that if $P_n(z) \in K[n, 1]$ and $Q_m(z) \in K[m, 1+\rho]$ $(\rho > 0)$, then

$$|R(re^{i\alpha})| \geq \left(\frac{1+r}{2}\right)^n \left(\frac{\rho}{1-r+\rho}\right)^m |R(e^{i\alpha})|, \tag{4}$$

where $r \leq 1$.

In the present article all the results listed above are sharpened in a corresponding way, and also, for the given class of polynomials, an inequality is obtained which is an analogue of the classical inequality of A. A. Markov ([5])

$$\max_{-1 \leq x \leq 1} |P_n'(x)| \leq n^2 M. \tag{5}$$

and S. N. Bernstein' s inequality (7)

$$|P_n'(x)| \leq \frac{nM}{\sqrt{1-x^2}}. \tag{5'}$$

Theorem 1. Let $P_n(z)$ be of the class $K[n, \sqrt{R}+\gamma]$, $R \geq 1$, $\gamma > 0$; let $z_j = r_j e^{i\varphi_j}$ be the zeros of this polynomial. Then for $r \leq 1$ the inequality

$$\left|P_n\left(Re^{i\omega}\right)\right| \leq \left(\frac{R+d}{r+d}\right)^n \left|P_n\left(re^{i\psi}\right)\right| \tag{6}$$

holds under the condition

$$\cos(\psi - \varphi_j) \leq \cos(\omega - \varphi_j), \qquad d = \min r_j \quad (j = 1, 2, \ldots, n).$$

Corollary. Let the polynomial $P_n(z)$ be of the class $K[n, 1]$. Then for the trigonometric polynomial $P_n(e^{i\theta}) = T_n(\theta)$ of degree n the inequality

$$|T_n(\omega)| \leq |T_n(\psi)|$$

holds under the condition

$$\cos(\psi - \varphi_j) \leq \cos(\omega - \varphi_j).$$

Theorem 2. Let $P_n(z)$ be of the class $K[n, R+\rho]$, where $R \geq 1$, $\rho > 0$, and let $z_j = r_j e^{i\varphi_j}$ be the zeros of this polynomial. Then for $r \leq 1$ the inequality

$$\left|P_n\left(Re^{i\omega}\right)\right| \geq \left(\frac{\sigma - R}{\sigma - r}\right)^n \left|P_n\left(re^{i\psi}\right)\right| \tag{7}$$

holds under the condition

$$\cos(\psi - \varphi_j) \leq \cos(\omega - \varphi_j), \qquad \sigma = \min r_j \quad (j = 1, 2, \dots, n).$$

Let $P_n(z) \in K[n, \sqrt{R}]$; then inequality (7) is true if in it the moduli of the polynomials are replaced by the maximum of their moduli.

Theorem 3. Let $P_n(z)$ be of the class $K[n, \sqrt{r'} + \gamma]$, where $r' \geq 1$, $\gamma > 0$; let $z_j = r_j e^{i\varphi_j}$ $(j = 1, 2, \dots, n)$ be the zeros of this polynomial, and let $Q_m(z) \in K[m, r' + \rho]$, where $r' \geq 1$, $\rho > 0$, and $\tau_k = \rho_k e^{i\alpha_k}$ $(k = 1, 2, \dots, m)$ be the zeros of $Q_m(z)$. Then for $r \leq 1$ the inequality

$$\left|R\left(r'e^{i\omega}\right)\right| \leq \left(\frac{r' + d}{r + d}\right)^n \left(\frac{\sigma - r'}{\sigma - r}\right)^m \left|R\left(re^{i\psi}\right)\right| \tag{8}$$

holds under the conditions that

$$\cos(\psi - \varphi_j) \leq \cos(\omega - \varphi_j), \qquad \cos(\psi - \alpha_k) \leq \cos(\omega - \alpha_k),$$

$$d = \min r_j, \qquad \sigma = \min \rho_k.$$

Remark. Inequalities (6) and (8) are sharp. Indeed, in paper (5) polynomials and rational functions for which equality is attained are given for a special case.

Theorem 4*. Let $P_n(z)$ be an arbitrary polynomial of degree n; let $Q_m(z)$ be a polynomial of the class $K[m, \rho + \gamma]$, $\rho \geq 1$, $\gamma > 0$; and let $z_j = r_j e^{i\varphi_j}$ $(j = 1, 2, \dots, m)$ be the zeros of this polynomial.

If

$$|R(z)| = \left|\frac{P_n(z)}{Q_m(z)}\right| \leq L \quad \text{for } |z| \leq 1,$$

then

$$|R(\rho e^{i\alpha})| \leq \rho^n \left(\frac{\sigma - 1}{\sigma - \rho}\right)^m L, \tag{9}$$

where

$$\sigma = \min r_j \quad (j = 1, 2, \dots, m).$$

Remark. Under the conditions of Theorem 4, inequality (9) can also be written in the form

$$|R(\rho e^{i\alpha})| \le \left(\frac{\rho}{r}\right)^n \left(\frac{\sigma - r}{\sigma - \rho}\right)^m \max_{|z|=r} |R(re^{i\alpha})|, \tag{9'}$$

where $r \le 1$.

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* This theorem is a refinement of Walsh' s inequality ((8), p. 282).

We shall give two more theorems analogous to Theorem 1.

Theorem 5. *Let $P_n(z) \in K[n, 1]$; $z_j = r_j e^{i\varphi_j}$ $(j = 1, 2, \dots, n)$ be the zeros of this polynomial. Then for $x \leqslant 1$ the inequality*

$$\left|\rho_n(e^{i\alpha})\right| \leqslant \left(\frac{1+d}{|x|+d}\right)^n |P_n(x)| \tag{10}$$

holds, provided that $\cos\varphi_j \leqslant \cos(\alpha - \varphi_j)$, $d = \min r_j$ $(j = 1, 2, \dots, n)$.

If in the right-hand side of inequality (10) the modulus of the polynomial is replaced by the maximum of the modulus, then $P_n(z)$ may be taken from the class $K[n, \sqrt{|x|}]$.

Theorem 6. *Let $P_n(z)$ be any polynomial of degree n; $z_j = r_j e^{i\varphi_j}$, $(j = 1, 2, \dots, n)$, be the zeros of this polynomial. If $|P_n(x_0 e^{i\alpha_0})| = \max_{|z| \leqslant x_0} |P_n(z)|$, then for $x_0 \leqslant 1$ the inequality*

$$|P_n(x_0 e^{i\alpha_0})| \leqslant |P_n(x_0)| \tag{11}$$

holds, provided that $\cos\varphi_j \leqslant \cos(\alpha_0 - \varphi_j)$ $(j = 1, 2, \dots, n)$.

We now give an inequality which is an analogue of the inequalities of A. A. Markov (5) and S. N. Bernstein (5′) for the given class.

Theorem 7*. *Under the assumptions of Theorem 6, for any polynomial $P_n(z)$ the inequality*

$$|P_n'(x)| \leqslant n \max_{-1 \leqslant x \leqslant 1} |P_n(x)|. \tag{12}$$

holds.

If $P_n(z) \in K[n,1]$ and the assumptions of Theorem 6 are satisfied, then

$$|P_n'(x)| \leqslant n/2 \tag{13}$$

provided that $|P_n(x)| \leqslant 1$ on $[-1,1]$.

It is not difficult to observe that analogous assertions hold in the metric of the space $\mathcal{L}_p$.

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* Erdős conjectured and Lax proved in (9) that if $P(z) \in K[n,1]$ and $|P_n(z)| \leqslant 1$ for $|z| \leqslant 1$, then $|P_n'(z)| \leqslant n/2$.

Note: Figure translations are in progress. See original paper for figures.