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Abstract

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ON A MODIFIED PROBLEM T_1 FOR THE EQUATION

$$u_{xx} + \operatorname{sign} y \, u_{yy} = 0$$

(Presented by Academician M. A. Lavrent' ev on 29 XII 1962)

In the present note we consider* a modified problem T_1 for the Lavrent' ev-Bitsadze equation

$$u_{xx} + \operatorname{sign} y \, u_{yy} = 0. \tag{1}$$

Let D be a simply connected domain of the xy-plane, bounded by a Jordan curve σ with endpoints at the points $A(-1, 0)$, $B(1, 0)$, situated in the upper half-plane $y > 0$, and by the characteristics $AC : y = -x - 1$ and $BC : y = x - 1$, issuing from the point $C(0, -1)$. Let $E_k(a_k, 0)$, $k = 1, \dots, n$, $-1 < a_1 < \dots < a_n < 1$, be prescribed points of the segment AB. The points

$$A_k \left[\frac{1}{2}(a_k - 1), -\frac{1}{2}(a_k + 1)\right] \quad \text{and} \quad B_k \left[\frac{1}{2}(a_k + 1), \frac{1}{2}(a_k - 1)\right],$$

$k = 0, 1, \dots, n + 1$ ($a_0 = -1$, $a_{n+1} = 1$), lie respectively on the characteristics AC and BC. Denote by D_1 and D_2 respectively the elliptic and hyperbolic parts of the mixed domain D.

Modified problem T_1. It is required to determine a function $u(x, y)$ with the following properties: 1) $u(x, y)$ is a solution of equation (1) in the domain D everywhere except for the points of the segment AB, the real axis, and the characteristics $E_{kA}k$, $E_{kB}k$; 2) $u(x, y)$ is continuous in the closed domain $\overline{D}$; 3) the partial derivatives u_x and u_y are continuously matched at all points of the segment AB, except possibly the points E_k, $k = 0, 1, \dots, n + 1$, at which u_x and u_y may become infinite of order less than one; 4) $u(x, y)$ assumes the prescribed values

$$u = \varphi \quad \text{on } \sigma; \tag{2}$$

$$u = \psi_k \text{ on } A_{kA_{k+1}} \text{ for even } k; \qquad u = \psi_k + \alpha_k \text{ on } B_{kB_{k+1}} \text{ for odd } k, \tag{3}$$

where φ is continuous, while $\psi_k(x)$, $k = 0, 1, \dots, n$, are twice differentiable functions whose second derivatives satisfy the Hölder condition, with $\psi_0(-1) = \varphi(-1)$; α_k are real constants not prescribed in advance.

Let $n = 2m$. The case $n = 2m - 1$ is investigated analogously. The modified problem T_1 cannot have more than one solution. This assertion reduces to the fact that if a solution $u(x, y)$ of the modified problem T_1 assumes the values $u = 0$ on σ, $u = 0$ on $A_{2k}A_{2k+1}$, $k = 0, 1, \dots, m$, $u = \alpha_{2k-1}$ on $B_{2k-1}B_{2k}$, $k = 1, \dots, m$, then necessarily $u = 0$ in the domain D and $\alpha_{2k-1} = 0$, $k = 1, \dots, m$. In the domain D_2 the function $u(x, y)$ has the form

$$u(x, y) = \frac{\tau(x + y) + \tau(x - y)}{2} + \frac{1}{2} \int_{x-y}^{x+y} \nu(t)\, dt, \tag{4}$$

where $\tau(x) = u(x, 0)$, $-1 \le x \le 1$, $\nu(x) = u_y(x, 0)$, $-1 < x < 1$.

By virtue of (3), from (4) we obtain

$$u_x - \lambda(x) u_y = f(x), \qquad y = 0, \qquad a_k < x < a_{k+1}, \quad k = 0, 1, \dots, 2m, \tag{5}$$

where $\lambda(x) = -1$ on L_1; $\lambda(x) = 1$ on L_2; $f(x) = \psi'_{2k-1}[\frac{1}{2}(x + 1)]$ on L_1; $f(x) = \psi_{2k}[{}^1/{}_2(x - 1)]$ on L_2; L_1 and L_2 denote respectively the union of the intervals (a_{2k-1}, a_{2k}), $k = 1, \dots, m$, and (a_{2k}, a_{2k+1}), $k = 0, 1, \dots, m$.

Hence, just as in problem T_1, we conclude that if $\psi_k(x) \equiv 0$, $k = 0, 1, \dots, 2m$, then the solution $u(x, y)$ of the modified problem T_1 in the closed domain $\overline{D}_1$ cannot attain a nonzero extremum in the intervals $a_k < x < a_{k+1}$, $k = 0, 1, \dots, 2m$, of the segment AB. It is easy to see that the solution $u(x, y)$ cannot have a nonzero extremum also at the points E_k, $k = 1, \dots, 2m$. A known property of the characteristic quadrilateral for the string equation allows one to assert that $u(a_{2k-1}, 0) = u(a_{2k}, 0) = \alpha_{2k-1}$, $k = 1, \dots, m$. Suppose that the function $u(x, y)$ attains a nonzero extremum at some point E_{2k}, for example at the point E_{2k_0}. Then the value of the function $u(x, y)$ at the point E_{2k_0-1} will also be extremal. Separate* the point E_{2k_0-1} from the remaining points E_k by a level line $\Gamma : u(x, y) = \text{const}$, with endpoints on the segment AB and lying entirely in D_1. To the domain bounded by the line Γ and a segment of the real axis, we apply Green' s formula:

$$\iint (u_x^2 + u_y^2)\, dx\, dy = -\int (u - \text{const}) \frac{\partial u}{\partial n}\, ds,$$

where n is the inward normal. From this formula, by virtue of the equalities $u_x + u_y = 0$ on L_1 and $u_x - u_y = 0$ on L_2, we conclude that $u(x, y) = \text{const}$ throughout the domain D_1, and this is impossible when $\varphi \not\equiv 0$.

Consequently, if $\psi_k(x) \equiv 0$, $k = 0, 1, \dots, 2m$, the solution $u(x, y)$ of the modified problem T_1 in the closed domain $\overline{D}_1$ attains a nonzero extremum on the arc σ

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* If no additional restrictions are imposed on the functions ψ_k, then problem T_1 may fail to have a solution of the required form, a fact not indicated in works ($^{1-8}$).

(the extremum principle). From this principle the uniqueness of the solution of the modified problem T_1 follows immediately.

For simplicity we shall assume that σ coincides with the semicircle σ_0 with endpoints at the points A, B and $u = 0$ on σ_0. In addition, we shall assume that u_x and u_y are continuous in the closed domain $\overline{D}_1$ everywhere except, possibly, at the points E_k, $k = 0, 1, \dots, 2m+1$.

Denote by $\Phi(z)$ the function $u(x, y) + iv(x, y)$, holomorphic in the domain D_1 and satisfying the condition $\Phi(-1) = 0$. Analogously to problem T_1 (^{4}), the determination of the function $\Phi'(z)$ reduces to the determination of a function $\Phi'(z)$ that is piecewise holomorphic in the upper half-plane, has a zero of the second order at infinity, and satisfies the boundary conditions

$$\operatorname{Re}(1-i)\Phi'(x) = f(x) \quad \text{on } L_2, \qquad \operatorname{Im}(1-i)\Phi'(x) = -f(x) \quad \text{on } L_1, \tag{6}$$

$$\operatorname{Re}(1-i)\Phi'(x) = \frac{1}{x^2} f(1/x) \quad \text{on } \overline{L}_1, \qquad \operatorname{Im}(1-i)\Phi'(x) = -\frac{1}{x^2} f(1/x) \quad \text{on } \overline{L}_2,$$

where $\overline{L}_1$ and $\overline{L}_2$ denote respectively the union of the intervals (b_{2k}, b_{2k-1}), $k = 1, \dots, m$, and (b_{2k+1}, b_{2k}), $k = 0, 1, \dots, j-1, j+1, \dots, m$, $(-\infty, b_{2j})$, (b_{2j+1}, ∞), it being assumed that $a_{2j} < 0 < a_{2j+1}$; $b_k = 1/a_k$.

The solution of this problem of class h_0 is given by the Keldysh-Sedov formula (^{5,6})

$$(1-i)\Phi'(z) = \frac{1}{\pi i} \frac{R_1(z)}{R_2(z)} \int_{-\infty}^{\infty} \frac{R_2(t)}{R_1(t)} \frac{g(t)}{t-z}\, dt + \frac{C_0 + C_1 z + \dots + C_{2m-1} z^{2m-1}}{R(z)}, \tag{7}$$

where $g(x) = f(x)$ on L_2; $g(x) = -if(x)$ on L_1; $x^2 g(x) = f(1/x)$ on $\overline{L}_1$;

$$x^2 g(x) = -if(1/x) \quad \text{on } \overline{L}_2$$

and

$$R_1(z) = \left[(z+1) \prod_1^m (z - a_{2k})(z - b_{2k}) \right]^{1/2},$$

$$R_2(z) = \left[(z-1) \prod_1^m (z - a_{2k-1})(z - b_{2k-1}) \right]^{1/2}, \qquad R(z) = \left[(z^2 - 1) \prod_1^{2m} (z - a_k) \times \right.$$

$$\times (z - b_k)]^{1/2},$$

where by $R_1(z)/R_2(z)$ is meant the branch holomorphic on the plane cut along L_2, $\overline{L}_1$, taking the value 1 at infinity, and by $R(z)$ the branch, holomorphic on the plane cut in the same way, taking positive values on Ox for $x > b_{2j+1}$; $C_0, C_1, \dots, C_{2m-1}$ are arbitrary real constants.

We find the function $\Phi(z)$ by the formula $\Phi(z) = \int_{-1}^{z} \Phi'(\xi)\, d\xi$, and it is necessary that $\Phi(z)$ satisfy the condition $\Phi(1/\overline{z}) = -\Phi(z)$ (^{4}). It is easy to see that, in order for this condition to be fulfilled, it is necessary and sufficient that $C_k = C_{2m-k-1}$, $k = 0, 1, \dots, m-1$. To determine C_k and α_{2k+1}, $k = 0, 1, \dots, m-1$, we have the following conditions:

$$\operatorname{Re}\Phi(a_{2k+1}) = \psi_{2k}\left[\frac{1}{2}(a_{2k+1} - 1)\right] + \psi_{2k+1}\left[\frac{1}{2}(a_{2k+1} + 1)\right] - \psi_{2m}(0) + \alpha_{2k+1}, \qquad k = 0, 1, \dots, m-1,$$

$$\operatorname{Re}\Phi(a_{2k+2}) = \psi_{2k+2}\left[\frac{1}{2}(a_{2k+2} - 1)\right] + \psi_{2k+1}\left[\frac{1}{2}(a_{2k+2} + 1)\right] - \psi_{2m}(0) + \alpha_{2k+1}, \qquad k = 0, 1, \dots, m-1. \tag{8}$$

These conditions constitute a system of $2m$ linear equations with respect to C_k and α_{2k+1}, $k = 0, 1, \dots, m-1$:

$$\sum_{j=0}^{m-1} \beta_{kj} C_j - \alpha_{2k+1} = \beta_k, \qquad k = 0, 1, \dots, m-1,$$

$$\sum_{j=0}^{m-1} \gamma_{kj} C_j - \alpha_{2k+1} = \gamma_k, \qquad k = 0, 1, \dots, m-1, \tag{9}$$

where β_{kj} and γ_{kj} do not depend on $\psi_k(x)$, while $\beta_k = 0$ and $\gamma_k = 0$ when $\psi_k(x) \equiv 0$.

From the uniqueness of the solution of the modified problem T_1 it follows directly that system (9) is uniquely solvable.

The real part of the function $\Phi(z)$ gives the required function $u(x, y)$ in the domain D_1. In the domain D_2, the solution $u(x, y)$ is constructed by a known method.

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* If one separates the point E_{2k_0} by a level line, then one cannot conclude from Green' s formula that $u(x, y) = \text{const}$ in D_1.

Remark. In the case $n = 2m - 1,\ m > 1$, the values of the function $u(x, y)$ at the points $E_k,\ k = 1, \dots, 2m - 1$, are known up to a constant equal to the value of $u(x, y)$ at the point $C(0, -1)$. Denote $u(0, -1) = \delta$. To determine $C_k,\ k = 0, 1, \dots, m - 2$ (in this case $C_k = -C_{2m-k-2}$, $k = 0, 1, \dots, m - 2$, and $C_{m-1} = 0$), α_{2k-1}, $k = 1, \dots, m - 1$, and δ, one obtains a uniquely solvable system of $2m - 1$ linear equations with respect to C_k, α_{2k-1}, and δ.

For $n = 1$ the modified problem T_1 coincides with problem T_1. In this case problem T_1 is well posed (from the condition $\Phi(1/\overline{z}) = -\Phi(z)$ it follows that the only arbitrary constant C_0 appearing in the expression for $\Phi(z)$ is equal to zero).

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