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**Abstract**

**Full Text**

## **Reports of the Academy of Sciences of the USSR**

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### **MECHANICS OF CONTINUA**

**V. V. Lokhin**

## **GENERAL FORMS OF RELATION BETWEEN TENSOR FIELDS IN AN ANISOTROPIC CONTINUUM WHOSE PROPERTIES ARE DESCRIBED BY VECTORS, SECOND-RANK TENSORS, AND ANTISYMMETRIC THIRD-RANK TENSORS**

*(Presented by Academician L. I. Sedov on 18 XII 1962)*

We shall consider, in particular, the problem of the structure of nonlinear relations for tensors that characterize the physical properties of textures and crystals of lower syngonies.

We assume <sup>(1)</sup> that the model of an infinitesimal particle of a continuum is described by a finite set of characteristics—a system of defining parameters  $\mathfrak{G}_{(k)}$  ( $k = 1, 2, \dots, r$ ),  $\mathfrak{M}_{(l)}$  ( $l = 1, 2, \dots, s$ ),  $\mathfrak{P}_{(m)}$  ( $m = 1, 2, \dots, t$ ). The first of these (variable or constant) quantities  $\mathfrak{G}_{(k)}$  are geometric characteristics (describing the anisotropy of the medium), the second  $\mathfrak{M}_{(l)}$  are mechanical ones (velocity, density, deformation and rotation of the particle, etc.), and the remaining  $\mathfrak{P}_{(m)}$  are physical or chemical characteristics of the medium (for example, temperature, chemical composition of the substance, polarization and magnetization, coefficients of viscosity, thermal conductivity, electrical conductivity, etc.).

As is known, usually the system of equations of motion and state of a continuum is given as a set of invariant tensor relations between the unknown functions and the elements of the defining basis. In order to reveal the structure of these tensor relations, let us consider one of the tensor functions describing the state or motion of the continuum:

$$T^{i_1 i_2 \dots i_n} = T^{i_1 i_2 \dots i_n}(\mathfrak{G}_{(k)}, \mathfrak{M}_{(l)}, \mathfrak{P}_{(m)}).$$

Let  $\xi_{i_1}^{(1)}, \xi_{i_2}^{(2)}, \dots, \xi_{i_n}^{(n)}$  be the components of arbitrary  $n$  vectors  $\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(n)}$ .

The function

$$T^{i_1 i_2 \dots i_n} \xi_{i_1}^{(1)} \xi_{i_2}^{(2)} \dots \xi_{i_n}^{(n)}$$

which is multilinear in the components of the vectors  $\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(n)}$ , is a scalar function of the invariants and joint invariants of the elements of the defining basis and the vectors  $\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(n)}$ . The function  $\varphi$  evidently has the form

$$\varphi = \sum_s k_s(\mathfrak{G}_{(k)}, \mathfrak{M}_{(l)}, \mathfrak{P}_{(m)}) J_s(\mathfrak{G}_{(k)}, \mathfrak{M}_{(l)}, \mathfrak{P}_{(m)}, \xi^{(1)}, \xi^{(2)}, \dots, \xi^{(n)}),$$

where  $k_s$  are functions of the system of invariants of the defining parameters  $\{\mathfrak{G}_{(k)}, \mathfrak{M}_{(l)}, \mathfrak{P}_{(m)}\}$ , and  $J_s$  are products of joint invariants of the elements of the defining basis and the vectors  $\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(n)}$ , and of the invariants of the vectors  $\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(n)}$ , linear in the components of all the vectors  $\xi_{i_1}^{(1)}, \xi_{i_2}^{(2)}, \dots, \xi_{i_n}^{(n)}$ .

Because of the arbitrariness of the vectors  $\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(n)}$ , we have <sup>(7)</sup>

$$T^{i_1 i_2 \dots i_n} = \sum_s k_s \frac{\partial^n J_s}{\partial \xi_{i_1}^{(1)} \partial \xi_{i_2}^{(2)} \dots \partial \xi_{i_n}^{(n)}}. \quad (1)$$

We shall call *basis invariants* the set of functionally independent invariants, composed of the components of the tensors forming the system of defining parameters, such that the values of the independent com-

\* Some of them coincide if the tensor  $T$  possesses internal symmetry <sup>(3)</sup>.

of all tensors entering into the defining basis are uniquely expressed in terms of the values of invariants from this set. The choice of basis invariants is, evidently, not unique.

We may now suppose that the scalar coefficients  $k_s$  occurring in formula (1) depend in an arbitrary way on the basis invariants. It is clear that the maximum number of linearly independent terms on the right-hand side of formula (1) corresponds to the number of linearly independent components of the tensor\*.

Formula (1) reveals the structure of an arbitrary nonlinear tensor function describing the state or motion of a continuum.

Let the system of defining parameters, consisting of a finite number of quantities describing the state and motion of a continuum, include only: 1) vectors  $\overset{(n)}{b} = b^i \partial_i$ , ( $n = 1, 2, \dots$ ); 2) symmetric tensors of second rank  $\overset{(m)}{\mathcal{G}} = g^{ij} \partial_i \partial_j$ ,  $\overset{(m)}{\mathcal{E}} = \varepsilon_{ij}^{(m)} \partial^i \partial^j$ ,  $\varepsilon^{ij} = \varepsilon^{ji}$  ( $m = 1, 2, \dots$ ); 3) antisymmetric tensors of second rank  $\overset{(r)}{\Omega} = \Omega^{ij} \partial_i \partial_j$ ,  $\Omega^{ij} = -\Omega^{ji}$  ( $r = 1, 2, \dots$ ); 4) antisymmetric tensors of third rank

$$e = \Delta \varepsilon^{ijk} \partial_i \partial_j \partial_k = \Delta [\partial_1 (\partial_2 \partial_3 - \partial_3 \partial_2) + \partial_2 (\partial_3 \partial_1 - \partial_3 \partial_1) + \partial_3 (\partial_2 \partial_1 - \partial_1 \partial_2)],$$

where

$$\overset{(s)}{\Pi} = \pi^{ijk} \partial_i \partial_j \partial_k, \quad \pi^{ijk} = -\pi^{ikj} \quad (s = 1, 2, \dots),$$

where  $\partial_i$  are the unit vectors of the coordinate system;  $\Delta = \pm 1/\sqrt{g}$ ,  $\sqrt{g} = \text{mod} |(\partial_i \partial_j)|$ , and the plus sign corresponds to a right-handed, and the minus sign to a left-handed, coordinate system.

The tensor of third rank  $e = \Delta \varepsilon^{ijk} \partial_i \partial_j \partial_k$  is invariant with respect to the proper orthogonal group of coordinate transformations, i.e. with respect to all rotations of space about a fixed point (without reflection). The components of the tensor  $e$  are determined by the equalities:

$$e^{ijk} = \Delta \varepsilon^{ijk} = \begin{cases} 0, & \text{if } i = j \text{ or } i = k, \\ \Delta, & \text{if the permutation } \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix} \text{ is even,} \\ -\Delta, & \text{if the permutation } \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix} \text{ is odd.} \end{cases}$$

Defining parameters of precisely this type characterize the geometric properties (anisotropy) of various textures and crystals of the lower systems.

Indeed, as defining parameters characterizing the anisotropy of a medium, one may introduce the following quantities <sup>(5)</sup>:

For textures

1. Class  $K_h(\infty/\infty \cdot m)$ :  $I = \mathbf{e}_1 \mathbf{e}_1 + \mathbf{e}_2 \mathbf{e}_2 + \mathbf{e}_3 \mathbf{e}_3$ .
2. Class  $K(\infty/\infty)$ :  $I, E = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 - \mathbf{e}_1 \mathbf{e}_3 \mathbf{e}_2 + \mathbf{e}_2 \mathbf{e}_3 \mathbf{e}_1 - \mathbf{e}_2 \mathbf{e}_1 \mathbf{e}_3 + \mathbf{e}_3 \mathbf{e}_1 \mathbf{e}_2 - \mathbf{e}_3 \mathbf{e}_2 \mathbf{e}_1$ .
3. Class  $C_{\infty v}(\infty \cdot m)$ :  $I, \mathbf{e}_1$ .
4. Class  $D_{\infty h}(m \cdot \infty : m)$ :  $I, \mathbf{e}_1 \mathbf{e}_1$ .
5. Class  $C_{\infty}(\infty)$ :  $I, \mathbf{e}_1, \Omega = \mathbf{e}_3 \mathbf{e}_2 - \mathbf{e}_2 \mathbf{e}_3$ .
6. Class  $C_{\infty h}(\infty : m)$ :  $I, \mathbf{e}_1 \mathbf{e}_1, \Omega$ .
7. Class  $D_{\infty}(\infty : 2)$ :  $I, \mathbf{e}_1 \mathbf{e}_1, E$ .

For crystals

a) Triclinic system

8. Class  $C_1(1)$ :  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ .

9. Class  $C_i(\bar{2})$ :  $\mathbf{e}_1\mathbf{e}_1, \mathbf{e}_2\mathbf{e}_2, \mathbf{e}_3\mathbf{e}_3, \mathbf{e}_3\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_1\mathbf{e}_3 - \mathbf{e}_3\mathbf{e}_1, \mathbf{e}_2\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_2$ .

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\* By examining the right-hand side of formula (1), one can find the group  $G$  of internal and external symmetry<sup>(3)</sup> of the tensor  $T$ . A theorem from the theory of group representations (the character theorem) makes it possible to compute the number of linearly independent tensors of a given rank that are invariant with respect to the group  $G$ , i.e. the number of linearly independent components of the tensor  $T$ . The number of linearly independent components  $p$  of the tensor  $T$  for different symmetry cases can be determined from tables of dimensions of group tensor spaces<sup>(3,4)</sup>.

**b) Monoclinic syngony**

10. Class  $C_s(m)$ :  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\mathbf{e}_3$ .

11. Class  $C_2(2)$ :  $\mathbf{e}_3, \mathbf{e}_1\mathbf{e}_1, \mathbf{e}_2\mathbf{e}_2, \mathbf{e}_2\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_2$ .

12. Class  $C_{2h}(2:m)$ :  $\mathbf{e}_1\mathbf{e}_1, \mathbf{e}_2\mathbf{e}_2, \mathbf{e}_3\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_2$ .

**c) Orthorhombic syngony**

13. Class  $C_{2v}(2 \cdot m)$ :  $\mathbf{e}_3, \mathbf{e}_1\mathbf{e}_1, \mathbf{e}_2\mathbf{e}_2$ .

14. Class  $D_2(2:2)$ :  $\mathbf{e}_1\mathbf{e}_1, \mathbf{e}_2\mathbf{e}_2, \mathbf{e}_3\mathbf{e}_3, E$ .

15. Class  $D_{2h}(m \cdot 2:m)$ :  $\mathbf{e}_1\mathbf{e}_1, \mathbf{e}_2\mathbf{e}_2, \mathbf{e}_3\mathbf{e}_3$ .

The vectors  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  are unit vectors of the crystallophysical coordinate system<sup>(6)</sup>. Products of the vectors  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  are polyadic products.

The list of determining quantities given above follows from the data of works<sup>(2,7)</sup>. We have retained only independent and essential parameters<sup>(5)</sup>.

Using the results of works<sup>(8,9)</sup>, one can write out the basis invariants and find the structure of arbitrary invariant tensor functions of an arbitrary number of vectors, symmetric and antisymmetric tensors of second rank, and antisymmetric tensors of third rank.

Let us note that, by virtue of the duality relations between antisymmetric tensors (absolute or relative) and relative (sometimes weight) tensors of rank lower by one, the problem posed reduces to finding tensor functions invariant with respect to the proper orthogonal group of coordinate transformations, depending on vectors and symmetric tensors of second rank. Let us also note that the

symmetry groups of the most general tensor functions in this case are easily determined as the intersection of the symmetry groups of the totality of arguments (the Neumann–Curie principle <sup>(10, 11)</sup>).

Knowing the symmetry group of a tensor function, from the tables of dimensions of group tensor spaces <sup>(3, 4)</sup> one can find the maximum number  $p$  of linearly independent terms that should be retained in the right-hand side of the decomposition found (see formula (1)). Discarding superfluous terms, we obtain formulas which are, in essence, a generalization of the Hamilton–Cayley formula (see <sup>(1)</sup>).

Let us illustrate what has been said by the example of textures.

**Example 1.** The most general scalar, vector, and tensor functions (of second, third, and fourth ranks) of the scalars  $\beta_{(n)}$  ( $n = 1, 2, \dots, N$ ) and the fundamental tensor  $\mathcal{G} = g^{ij}\partial_i\partial_j$  have the form:  $\alpha = k(\beta_{(n)})$ ,  $a^i = 0$ ,  $A^{ij} = kg^{ij}$ ,  $A^{ij\alpha} = 0$ ,

$$A^{ij\alpha\beta} = k_1g^{ij}g^{\alpha\beta} + k_2g^{i\alpha}g^{j\beta} + k_3g^{i\beta}g^{j\alpha}.$$

The coefficients  $k$  and  $k_i$  are arbitrary functions of the scalars  $\beta_{(n)}$ .

**Example 2.** Arbitrary functions of the scalars  $\beta_{(n)}$ , the metric tensor  $\mathcal{G} = g^{ij}\partial_i\partial_j$  and the vector  $b = b^i\partial_i$  have the form:  $\alpha = k(\beta_{(n)}, b^ib_i)$ ,

$$a^i = kb^i, \quad A^{ij} = k_1g^{ij} + k_2b^ib^j,$$

$$A^{ij\alpha} = k_1g^{ij}b^\alpha + k_2g^{i\alpha}b^j + k_3g^{j\alpha}b^i + k_4b^ib^jb^\alpha,$$

$$\begin{aligned} A^{ij\alpha\beta} = & k_1g^{ij}g^{\alpha\beta} + k_2g^{i\alpha}g^{j\beta} + k_3g^{i\beta}g^{j\alpha} + k_4g^{ij}b^\alpha b^\beta + k_5g^{i\alpha}b^jb^\beta + k_6g^{i\beta}b^jb^\alpha \\ & + k_7g^{\alpha\beta}b^ib^j + k_8g^{j\beta}b^ib^\alpha + k_9g^{j\alpha}b^ib^\beta + k_{10}b^ib^jb^\alpha b^\beta, \end{aligned}$$

where  $k$  and  $k_i$  are arbitrary functions of the scalars  $\beta_{(n)}$  and  $|b|^2 = b^ib_i$ .

**Example 3.** Tensor functions of the degenerate tensor of second rank  $B = B^{ij}\partial_i\partial_j = b^ib^j\partial_i\partial_j$ , the scalars  $\beta_{(n)}$ , and the metric tensor  $\mathcal{G} = g^{ij}\partial_i\partial_j$  are obtained as a special case from the functions of Example 2:

$$\alpha = k(\beta_{(n)}, B_i^i), \quad a^i = 0, \quad A^{ij} = k_1g^{ij} + k_2B^{ij}, \quad A^{ij\alpha} = 0,$$

$$\begin{aligned} A^{ij\alpha\beta} = & k_1g^{ij}g^{\alpha\beta} + k_2g^{i\alpha}g^{j\beta} + k_3g^{i\beta}g^{j\alpha} + k_4g^{ij}B^{\alpha\beta} + k_5g^{i\alpha}B^{j\beta} + k_6g^{i\beta}B^{j\alpha} \\ & + k_7g^{\alpha\beta}B^{ij} + k_8g^{j\beta}B^{i\alpha} + k_9g^{j\alpha}B^{i\beta} + k_{10}B^{ij}B^{\alpha\beta}, \end{aligned}$$

where  $k$  and  $k_i$  are arbitrary functions of the scalars  $\beta_{(n)}$  and the single invariant  $B_i^i$  of the tensor  $B = B^{ij}\partial_i\partial_j$ .

In Examples 1-3, the number of terms on the right-hand side of each formula corresponds—

coincides with the dimension of the corresponding tensor space. All terms in the indicated formulas are linearly independent.

**Example 4.** The general formulas for functions depending on an isotropic tensor of third rank  $E = E^{ijk}\partial_i\partial_j\partial_k = \omega\Delta e^{ijk}\partial_i\partial_j\partial_k$ , on scalars  $\beta_{(n)}$ , and on the fundamental tensor  $\mathcal{G} = g^{ij}\partial_i\partial_j$ , have the form:  $\alpha = k(\beta_{(n)}, \omega^2)$ ,  $a^i = 0$ ,  $A^{ij} = kg^{ij}$ ,  $A^{ij\alpha} = ke^{ij\alpha}$ ,  $A^{ij\alpha\beta} = k_1g^{ij}g^{\alpha\beta} + k_2g^{i\alpha}g^{j\beta} + k_3g^{i\beta}g^{j\alpha}$ , where  $k$  and  $k_i$  depend on  $\beta_{(n)}$  and  $\omega^2$ .

**Example 5.** The general form of the dependence on a distinguished tensor of second rank  $B = B^{ij}\partial_i\partial_j = b^ib^jg^2j$ , an isotropic tensor of third rank  $E = E^{ijk}\partial_i\partial_j\partial_k = \omega\Delta e^{ijk}\partial_i\partial_j\partial_k$ , and the metric tensor  $\mathcal{G} = g^{ij}\partial_i\partial_j$  is as follows:  $\alpha = k(\beta_{(n)}, B_i^i, \omega^2)$ ,  $a^i = 0$ ,  $A^{ij} = k_1g^{ij} + k_2B^{ij}$ ,  $A^{ij\alpha} = k_1e^{ij\alpha} + k_2B_\gamma^ie^{\gamma\alpha} + k_3e^{ij\gamma}B_\gamma^\alpha$ ,  $A^{ij\alpha\beta} = k_1g^{ij}g^{\alpha\beta} + k_2g^{i\alpha}g^{j\beta} + k_3g^{i\beta}g^{j\alpha} + k_4g^{ij}B^{\alpha\beta} + k_5g^{i\alpha}B^{j\beta} + k_6g^{i\beta}B^{j\alpha} + k_7g^{\alpha\beta}B^{ij} + k_8g^{j\beta}B^{i\alpha} + k_9g^{j\alpha}B^{i\beta} + k_{10}B^{ij}B^{\alpha\beta}$ , where  $k$  and  $k_i$  are functions of  $\beta_{(n)}$ ,  $B_i^i$ , and  $\omega^2$ .

**Example 6.** The most general functions of an antisymmetric tensor  $\Omega = \Omega^{ij}\partial_i\partial_j$ , a vector  $\mathbf{b} = b^i\partial_i$ , and the metric tensor  $\mathcal{G} = g^{ij}\partial_i\partial_j$ , under the condition that  $\mathbf{b} = b\mathbf{e}_1$ ,  $\Omega = \chi(\mathbf{e}_3\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_3)$ , where  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  are mutually perpendicular vectors, have the form:  $\alpha = k(\beta_{(n)}, b^ib_i, \Omega_{ij}\Omega^{ij})$ ,  $a^i = kb^i$ ,  $A^{ij} = k_1g^{ij} + k_2b^ib^j + k_3\Omega^{ij}$ ,  $A^{ij\alpha} = k_1g^{ij}b^\alpha + k_2g^{i\alpha}b^j + k_3g^{j\alpha}b^i + k_4\Omega^{ij}b^\alpha + k_5\Omega^{\alpha j}b^i + k_6\Omega^{ij}b^\alpha + k_7b^ib^jb^\alpha$ ,  $A^{ij\alpha\beta} = k_1g^{ij}g^{\alpha\beta} + k_2g^{i\alpha}g^{j\beta} + k_3g^{i\beta}g^{j\alpha} + k_4g^{ij}b^\alpha b^\beta + k_5g^{i\alpha}b^jb^\beta + k_6g^{i\beta}b^jb^\alpha + k_7g^{\alpha\beta}b^ib^j + k_8g^{j\beta}b^ib^\alpha + k_9g^{j\alpha}b^ib^\beta + k_{10}b^ib^jb^\alpha b^\beta + k_{11}g^{ij}\Omega^{\alpha\beta} + k_{12}g^{i\alpha}\Omega^{j\beta} + k_{13}g^{i\beta}\Omega^{j\alpha} + k_{14}g^{\alpha\beta}\Omega^{ij} + k_{15}g^{j\beta}\Omega^{i\alpha} + k_{16}g^{j\alpha}\Omega^{i\beta} + k_{17}\Omega^{ij}\Omega^{\alpha\beta} + k_{18}\Omega^{i\alpha}\Omega^{j\beta} + k_{19}\Omega^{i\beta}\Omega^{j\alpha}$ , where  $k$  and  $k_i$  are arbitrary functions of  $\beta_{(n)}$ ,  $b^ib_i$ ,  $\Omega_{ij}\Omega^{ij}$ .

**Example 7.** The general forms of the dependence on a distinguished tensor  $B^{ij}\partial_i\partial_j = b^ib^j\partial_i\partial_j$ , an antisymmetric tensor  $\Omega = \Omega^{ij}\partial_i\partial_j$  (see Example 6), scalars  $\beta_{(n)}$ , and the metric tensor  $\mathcal{G} = g^{ij}\partial_i\partial_j$  can be represented in the form:  $\alpha = k(\beta_{(n)}, B_i^i, \Omega^{ij}\Omega_{ij})$ ,  $a^i = 0$ ,  $A^{ij} = k_1g^{ij} + k_2B^{ij} + k_3\Omega^{ij}$ ,  $A^{ij\alpha} = 0$ ,  $A^{ij\alpha\beta} = k_1g^{ij}g^{\alpha\beta} + k_2g^{i\alpha}g^{j\beta} + k_3g^{i\beta}g^{j\alpha} + k_4g^{ij}B^{\alpha\beta} + k_5g^{i\alpha}B^{j\beta} + k_6g^{i\beta}B^{j\alpha} + k_7g^{\alpha\beta}B^{ij} + k_8g^{j\beta}B^{i\alpha} + k_9g^{j\alpha}B^{i\beta} + k_{10}B^{ij}B^{\alpha\beta} + k_{11}g^{ij}\Omega^{\alpha\beta} + k_{12}g^{i\alpha}\Omega^{j\beta} + k_{13}g^{i\beta}\Omega^{j\alpha} + k_{14}g^{\alpha\beta}\Omega^{ij} + k_{15}g^{j\beta}\Omega^{i\alpha} + k_{16}g^{j\alpha}\Omega^{i\beta} + k_{17}\Omega^{ij}\Omega^{\alpha\beta} + k_{18}\Omega^{i\alpha}\Omega^{j\beta} + k_{19}\Omega^{i\beta}\Omega^{j\alpha}$ , where  $k$  and  $k_i$  are arbitrary functions of  $\beta_{(n)}$ ,  $B_i^i$ , and  $\Omega^{ij}\Omega_{ij}$ .

The linear independence of the terms on the right-hand sides of the formulas in Examples 4-7 either follows directly, or can be verified by computing the corresponding determinant.

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