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Abstract

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MATHEMATICS

B. Efimov

METRIZABILITY AND THE Σ-PRODUCT OF BICOMPACTS

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§ 1. $\Sigma_{\mathfrak{m}}$-products of bicompacts. Let $X = \prod_{\alpha\in\Theta} X_\alpha$ be the product of topological spaces X_α, with $\operatorname{card}\Theta \geq \aleph_1$. Let $p = (p_\alpha) \in X$. Following Corson ([1]), by the Σ-product of the spaces X_α with base point p we shall mean the subset $\Sigma^p \subset X$ consisting of those points $x = (x_\alpha) \in X$ for which $x_\alpha \neq p_\alpha$ for at most a countable set of indices $A_x \subset \Theta$. Let $\mathfrak{m} \geq \aleph_0$. Then by the $\Sigma_{\mathfrak{m}}$-product of the spaces X_α with base point $p = (p_\alpha)$ we shall mean the subset $\Sigma^p_{\mathfrak{m}} \subset X$ consisting of those points $x = (x_\alpha) \in X$ for which $x_\alpha \neq p_\alpha$ for at most a set of indices of cardinality $\leq \mathfrak{m}$.

Theorem 1. The weight of a bicompactum that is a continuous image of the $\Sigma_{\mathfrak{m}}$-product of bicompacts X_α of weight $\mathfrak{m}_\alpha \geq 2$ does not exceed

$$\max\left(\mathfrak{m},\ \sup_\alpha \mathfrak{m}_\alpha\right).$$

In particular, if the weight of X_α is $\leq \aleph_0$ for all $\alpha \in \Theta$ and $\mathfrak{m} = \aleph_0$, then as a corollary we obtain a result partially generalizing Corson' s result ([1]), who proved that a metric space that is a continuous image of the Σ-product of complete separable metric spaces is separable.

Corollary. A bicompactum that is a continuous image of the Σ-product of compacta is metrizable.

With the aid of Theorem 1 the following basic result is proved.

Theorem 2. The weight of a bicompactum R that is a continuous image of the bicompactum

$$X = \prod_{\alpha\in\Theta} X_\alpha,$$

where X_α is a bicompactum of weight $\mathfrak{m}_\alpha \geq \aleph_0$, does not exceed

$$\max\left[\sup_\alpha \mathfrak{m}_\alpha,\ \sup_{x\in R}\chi(x,R)\right].$$

In particular, if $X = D^\tau$, $\tau \geq \aleph_1$ (D^τ is the generalized Cantor discontinuum of weight τ), then we obtain the well-known theorem that the weight of a dyadic bicompactum is equal to $\sup_{x\in R}\chi(x,R)$.*

Lemma 1. Let R be a bicompactum of weight $\geq \tau$. Then the following two assertions are valid: 1) R contains a bicompactum R' of density $\leq \tau$ and weight $\geq \tau$; 2) R maps onto a bicompactum R'' of weight τ.**

Proof. We note that 2) implies 1). Indeed, let $R'' = f(R)$, with the weight of R'' equal to τ. Let M be dense in R'' and $\operatorname{card} M \leq \tau$. For each $x \in M$ choose a point $y \in f^{-1}(x)$. Denote the set of all chosen points by L. It is easy to see that $\operatorname{card} L \leq \tau$ and that $f[L] = [M]_{R''} = R''$, and therefore the weight of $[L]_R \geq \tau$. Thus, $[L] = R'$ is the required one.

We now prove 2). Let $C(R)$ be the ring of all continuous real functions on R with metric

$$\rho(g,g') = \sup_{x\in R}|g(x) - g'(x)|.$$

We note,

* $\chi(x,R)$ denotes the local character of the point x relative to R.
** The density of R is the least cardinality of an everywhere dense set.

that the weight of $C(R)$ = weight $R \geq \tau$ ([2]). Let $\mathfrak{B} = \{f_\alpha\}$, $f_\alpha \in C(R)$, be a maximal family of functions such that for any $f_\alpha, f_\beta \in \mathfrak{B}$, $\rho(f_\alpha, f_\beta) \geq 1$. We shall show that $\operatorname{card}\mathfrak{B} \geq \tau$. Suppose that $\operatorname{card}\mathfrak{B} \leq \tau' < \tau$. Add to this family the function $f \equiv 1$ and consider the subring $C'(R)$ of the ring $C(R)$ formed from all possible polynomials with rational coefficients in the family $\mathfrak{B}$. Note that the family $C'(R)$ does not separate all points of R, since otherwise, by M. Stone' s theorem ([3]), one would have $[C'(R)] = C(R)$, but $\operatorname{card} C'(R) = \tau'$ and, consequently, the weight of $C(R) = \tau' < \tau$, for for a metric space the weight coincides with the density. Thus there exist two points $x, y \in R$ such that for every function $f \in C'(R)$, $f(x) = f(y)$. Now consider a function $g \in C(R)$ such that $g(x) = 1$ and $g(y) = -1$. It is easy to see that for every $f_\alpha \in \mathfrak{B}$, $\rho(f_\alpha, g) \geq 1$, which contradicts the maximality of the family $\mathfrak{B}$. Thus $\operatorname{card}\mathfrak{B} \geq \tau$. Without loss of generality we assume that $\operatorname{card}\mathfrak{B} = \tau$. On the other hand, to each function $f_\alpha \in C(R)$ we associate the interval $I_\alpha = f(R)$ and construct, by the method of A. N. Tikhonov ([4]), a topological embedding of R in $I^\tau = \prod_\alpha I_\alpha$. Now consider the face I^τ of the bicompactum I^τ

$$\prod_{\alpha} I_{\alpha},$$

which is $\prod_{\alpha} I_{\alpha}$, if I_{α} corresponds to $f_{\alpha} \in \mathfrak{B}$. Let $\mathfrak{F}$ be the natural projection of I^{τ} onto I^{τ}. We shall show that $\mathfrak{F}(R) = R''$ is the required bicompactum. Indeed, $R'' \subset I^{\tau}$; consequently, the weight of $R'' \leq \tau$, and therefore the density of $R'' \leq \tau$. However, in the ring $C(R'')$ there exists a family of functions $\mathfrak{B}$ of cardinality τ, therefore the weight of $C(R'') \geq \tau$, whence the weight of $R'' \geq \tau$. The lemma is proved.

Proof of Theorem 1. Suppose the contrary. Let $R = f(\Sigma^{p}_{\mathfrak{m}})$, with weight $R > \mathfrak{l} = \max(\mathfrak{m}, \sup_{\alpha} \mathfrak{m}_{\alpha})$. Denote by $\mathfrak{n}$ the least cardinal number greater than $\mathfrak{l}$. Then the weight of $R \geq \mathfrak{n}$. By Glicksberg' s theorem ([5]), the Stone-Čech compactification of Σ^{p} is X. Since $\Sigma^{p}_{\mathfrak{m}} \subset \Sigma^{p} \subset X$, we have $\beta\Sigma^{p}_{\mathfrak{m}} = \beta\Sigma^{p} = X$ (see ([6]), p. 89). Denote by φ the Stone extension of the mapping f of the bicompactum X onto R. Using Lemma 1, consider in R a bicompactum R' of density $\leq \mathfrak{n}$ and weight $\geq \mathfrak{n}$. Let M be a dense subset of R' of cardinality $\leq \mathfrak{n}$. For each point $x_{\nu} \in M$, choose a point $y^{\nu} \in f^{-1}(x_{\nu}) \in \Sigma^{p}_{\mathfrak{m}}$. Let A_{ν} be the set of all indices for which $y^{\nu}_{\alpha} \neq p_{\alpha}$, if $\alpha \in A_{\nu}$. Since $y^{\nu} \in \Sigma^{p}_{\mathfrak{m}}$, card $A_{\nu} \leq \mathfrak{m}$. Put $A = \bigcup_{\nu} A_{\nu}$; then card $A \leq \mathfrak{n}$, since $\mathfrak{m} < \mathfrak{n}$. Consider

$$X_0 = \prod_{\alpha \in A} X_{\alpha}.$$

This set may be regarded as a subspace of X containing all the points y^{ν}, by putting, for all $y = (y_{\alpha}) \in X_0$, $y_{\alpha} = p_{\alpha}$, if $\alpha \in \Theta \setminus A$. Further, since $2^{\mathfrak{l}} \geq \mathfrak{n}$ and the density of $X_{\alpha} \leq \mathfrak{m}_{\alpha} \leq \mathfrak{l}$ for all $\alpha \in A$, the density of $X_0 \leq \mathfrak{l}$ ([7]). Thus the bicompactum $\Phi = \varphi(X_0)$, first, has density $\leq \mathfrak{l}$, and, second, contains R', hence has weight $\geq \mathfrak{n}$. We shall show that this cannot be. Let L be a dense subset of Φ of cardinality $\leq \mathfrak{l}$. For each $a_{\nu} \in L$, choose a point $b^{\nu} \in f^{-1}(a_{\nu}) \in \Sigma^{p}_{\mathfrak{m}}$. Let B_{ν} be the set of all indices for which $b^{\nu}_{\alpha} \neq p_{\alpha}$, if $\alpha \in B_{\nu}$. Since $b^{\nu} \in \Sigma_{\mathfrak{m}}$, card $B_{\nu} \leq \mathfrak{m}$. Put $B = \bigcup_{\nu} B_{\nu}$; then card $B \leq \mathfrak{l}$, since $\mathfrak{m} \leq \mathfrak{l}$. Consider

$$Y_0 = \prod_{\alpha \in B} X_{\alpha}.$$

As before, this set may be regarded as a subspace of X containing all the points b^{ν}, by putting, for all $y = (y_{\alpha}) \in Y_0$, $y_{\alpha} = p_{\alpha}$, if $\alpha \in \Theta \setminus B$. Note that Y_0 is a bicompactum of weight $\leq \mathfrak{l}$, since the weight of $X_{\alpha} \leq \mathfrak{m}_{\alpha} \leq \mathfrak{l}$ and card $B \leq \mathfrak{l}$. Therefore $F = [\{b^{\nu}\}]_X \subset Y_0$ and the weight of $F \leq \mathfrak{l}$. On the other hand,

$\varphi F = [L] = \Phi$, consequently, weight $\Phi \leq$ weight $F \leq \mathfrak{l}$, which contradicts the fact that weight $\Phi \geq \mathfrak{n}$. The theorem is proved.

Proof of Theorem 2. Suppose the contrary. Put

$$\mathfrak{l} = \max \left[\sup_{\alpha} \mathfrak{m}_\alpha,\ \sup_{x \in R} \chi(x, R) \right].$$

If $\mathfrak{n}$ is the least cardinal number greater than $\mathfrak{l}$, then weight $R \geq \mathfrak{n}$. Let $R = f(X)$ and let $\Sigma_{\mathfrak{l}}^p$ be the $\Sigma_{\mathfrak{l}}$-product of the X_α with base point p. Consider $Y = f(\Sigma_{\mathfrak{l}}^p)$. By Theorem 1, $Y \neq R$, for otherwise

$$\text{weight}\, R \leq \max \left(\mathfrak{l}, \sup_{\alpha} \mathfrak{m}_\alpha \right) \leq \mathfrak{l},$$

which contradicts the supposition. Let $x \in R \setminus Y$; then we shall show that $\chi(x, R) \geq \mathfrak{n} > \mathfrak{l}$, and the theorem will be proved. Indeed, if $\chi(x, R) \leq \mathfrak{l}$, then consider a fundamental system of neighborhoods $\{O_\nu x\}$ at the point x of cardinality $\mathfrak{l}$. Since Y is dense in R, choose for each point $z_\nu \in O_\nu x \cap Y$. Denote the set obtained by $Z = \{z_\nu\}$. Note that card $Z \leq \mathfrak{l}$ and that $[Z]_R \ni x$. We shall show that this cannot be. For each point $z_\nu \in Z$ choose a point $t^\nu \in f^{-1}(z_\nu) \cap \Sigma_{\mathfrak{l}}^p$. Denote by T_ν the set of all indices for which $t_\alpha^\nu \neq p_\alpha$; since $t^\nu \in \Sigma_{\mathfrak{l}}$, card $T_\nu \leq \mathfrak{l}$. Put

$$T = \bigcup_{\nu} T_\nu.$$

Consider

$$X_0 = \prod_{\alpha \in T} X_\alpha.$$

This set may be regarded as a subspace of X, containing all the points t^ν, by putting, for any $x = (x_\alpha) \in X_0$, $x_\alpha = p_\alpha$ if $\alpha \in \Theta \setminus T$. At the same time, since card $T \leq \mathfrak{l} \cdot \mathfrak{l} = \mathfrak{l}$, we have $X_0 \subset \Sigma_{\mathfrak{l}}^p$. On the other hand, X_0 is bicompact; therefore

$$[\{t^\nu\}]_X \subset X_0 \subset \Sigma_{\mathfrak{l}}^p.$$

Thus,

$$f[\{t^\nu\}]_X = [Z]_R \subset f(\Sigma_{\mathfrak{l}}^p) = Y,$$

which contradicts the fact that $[Z]_R \ni x$. The theorem is proved.

§ 2. **Metrizability criteria for dyadic bicompacts.**

Theorem 3. Let R be a dyadic bicompact. Then the following conditions are equivalent:

A. R is metrizable.
B. R is hereditarily normal.
C. The sequential closure of any subset in R coincides with the topological closure*.

Let us note that from this theorem it follows that a dyadic bicompact satisfying the first axiom of countability is metrizable, since in such a bicompact the sequential closure of any subset coincides with the topological one. On the other hand, combining Theorem 4 from the author' s paper ([8]) and Theorem 3 from the paper ([9]), we obtain another criterion for metrizability of dyadic bicompacts.

Theorem 4. Let R be a dyadic bicompact. Then the following conditions are equivalent:

D. R is metrizable.
E. R is hereditarily dyadic with respect to closed sets.
F. In R there exists everywhere dense subset with the first axiom of countability.

We shall need the following lemma, which is also of independent interest.

Lemma 2.** Every nonisolated point of a dyadic bicompact is countably attainable***.

Proof. Let x be a nonisolated point of a dyadic bicompact R. If $\chi(x, R) = \aleph_0$, then x is countably attainable. If $\chi(x, R) \geq \aleph_1$, then Theorem 4 ([8]) asserts that in R there lies $Y = b_0 E_{\aleph_1}$–

$*$ $x \in [M]$ in the sequential topology R, if there exists a countable sequence $[x_n] \in M$ converging to x.

$**$ As became known to me, an analogous result was obtained by M. Katetov.

$***$) A point x is countably attainable if in R there exists a countable sequence of pairwise distinct points converging to x.

the minimal bicompact extension of a discrete space of cardinality $\aleph_1$–so that the only non-isolated point of the bicompactum Y is the point x. However, it is easy to verify that the point x is countably attainable for the bicompactum Y. Consequently, x is countably attainable for R.

Proof of Theorem 3. Let us note that A follows from B and C; therefore, to prove the theorem it is sufficient to show that B implies A and C implies A.

1) **B implies A.** Let the weight $R \geqslant \aleph_1$; then $R = f(D^\tau)$, where $\tau \geqslant \aleph_1$. Let Σ^p be the corresponding Σ-product of simple two-point spaces lying in D^τ. Since R is not metrizable, it follows that $Y = f\Sigma^p \neq R$. Consider some point $x \in R \setminus Y$. We shall prove that $R \setminus x$ is not normal. Note that Σ^p is countably compact and, consequently, pseudocompact. It follows that Y is pseudocompact; hence $R \setminus x$, containing Y, is also pseudocompact. If $R \setminus x$ were normal, then it would be countably compact. But this is not so, since the point x is non-isolated and, consequently, countably attainable by Lemma 2.

2) **C implies A.** Suppose the contrary. Just as in the preceding proof, consider $Y = f(\Sigma^p)$, where $\Sigma^p \subset D^\tau$, $\tau \geqslant \aleph_1$. Since R is not metrizable, $Y \neq R$. Note that Y is dense in R and, consequently, $[Y]_R = R$. We now show that the closure of Y in the sequential topology coincides with

Y. Indeed, by virtue of the countable compactness of Y, no countable sequence $\{y_n\} \in Y$ can converge to a point $x \in R \setminus Y$. The theorem is proved.

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.