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V. D. REPNIKOV

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Abstract

Full Text

V. D. REPNIKOV

SOME THEOREMS ON THE STABILIZATION OF THE SOLUTION OF THE CAUCHY PROBLEM FOR PARABOLIC EQUATIONS

(Presented by Academician I. N. Vekua on VIII 1, 1962)

A general theorem on stabilization (the existence of a limit as $t \to \infty$) of the solution of the Cauchy problem for systems parabolic in the sense of I. G. Petrovskii, with coefficients depending only on time t, under the assumption that the initial function $u_0(x)$ has the so-called angular limiting means, was proved by S. D. Eidelman and F. O. Porper in ([1]). Recently Yu. N. Drozhzhin ([2]) established theorems in which, for linear parabolic equations of the second order with coefficients depending on t, the fact of stabilization was proved under the assumption that the initial function has spherical means. However, in this case either special restrictions are imposed on the sign of the initial function (and in this case the nature of the bodies for which the existence of the mean is assumed is largely immaterial), or it is assumed that as $t \to \infty$ the fundamental solution of the equation becomes spherically symmetric.

It appears that exact theorems on stabilization must necessarily take into account the properties of the equations themselves, as reflected in the properties of the fundamental solution of the Cauchy problem. The present note contains such theorems. As consequences of these theorems we obtain, in a certain sense, exact theorems on stabilization in the interesting case of equations of the second order.

The essential nature of the restrictions we impose is illustrated by an example.

1. Consider a one-parameter family of hypersurfaces (for fixed t, $t \geq t_0 \geq 0$) $\Phi_t(x) = c$ $(c \geq 0)$, possessing the following properties:

1°. It consists of closed simply connected hypersurfaces.

2°. If $r(c, \varphi, t)$ is the length of the vector joining the origin to a point of the hypersurface $\Phi_t(x) = c$ and making angles $\pi/2 - \varphi_i$ with the axes x_{i+1} $(i = 1, 2, \dots, n-1)$ of the Cartesian coordinate system, then $r(c, \varphi, t)$ is continuously positive with respect to the parameter c (the family of surfaces "inflates").

3°. The bodies $V^c_{\Phi_t}$, bounded by the hypersurfaces $\Phi_t(x) = c$, have a unique common point—the origin.

Definition 1. We shall say that the families $\Phi_t(x) = c$ tend to the family of closed hypersurfaces $F(x) = c$ as $t \to \infty$, if for any $\varepsilon > 0$ one can indicate such

SovietRxiv

a $T_0(\varepsilon)$ that for $t > T_0$ and any c

$$\frac{\operatorname{mes}\left(V_F^c \cup V_{\Phi_t}^c\right)}{\operatorname{mes}\left(V_F^c \cap V_{\Phi_t}^c\right)} < 1 + \varepsilon. \tag{1}$$

It is assumed that the limiting family of hypersurfaces $F(x) = c$ has properties $1^\circ, 2^\circ, 3^\circ$ and, in addition:

4°. For any $\varphi^{(1)}, \varphi^{(2)}$, and c, the inequality

$$k_0 r(c, \varphi^{(1)}) \le r(c, \varphi^{(2)}) \le k_1 r(c, \varphi^{(1)}) \tag{2}$$

holds (the letter k with a subscript will everywhere denote constants).

Definition 2. We shall call the ratio

$$S_F^c(u_0) = \int_{V_F^c} u_0(x)\, dx \,/\, \operatorname{mes} V_F^c$$

the mean of the measurable function $u_0(x)$ over the body V_F^c, and the limit of this ratio as $c \to \infty$ (if it exists) the limiting mean over the bodies V_F^c. We shall denote the limiting mean by the symbol $S_F^\infty(u_0)$.

2. We study the stabilization of the solution of the Cauchy problem for the parabolic system

$$\frac{\partial u}{\partial t} \sum_{|k| \le 2b} A_k(t) D^k u, \tag{3}$$

$$u(0, x) = u_0(x). \tag{4}$$

In what follows, everywhere $u_0(x)$ is assumed to be a bounded ($|u_0(x)| \le M$) measurable function.

Theorem 1. Suppose:

1) The Green' s matrix of system (3) satisfies condition ($\mathcal{L}_1^+$) ([1])

$$\left|D^{mG}(t, x)\right| \le k_2 a(t)^{-m-n} \exp\{-k_3(|x| a(t)^{-1})^q\} \tag{$\mathcal{L}_1^+$}$$

$$a(t) \to \infty \quad \text{as } t \to \infty.$$

2) The Green' s matrix is constant (in x) on the families $\Phi_t(x) = c$ (with properties $1^\circ, 2^\circ, 3^\circ$), which tend as $t \to \infty$ to the family $F(x) = c$ (with properties 1°–4°).

SovietRxiv

3) $A_0(t) \equiv 0$.

4) $u_0(x)$ has a limiting mean over the bodies V_F^c: $S_F^\infty(u_0) = l$.

Then the solution of problem (3), (4), represented by the Poisson integral, stabilizes to l uniformly in every finite domain of the space E_n.

Proof. First note the following: 1) since $A_0(t) \equiv 0$, one may assume $l = 0$; 2) from the existence of $S_F^\infty(u_0) = 0$ and (1) it follows that for any $\delta > 0$ one can indicate such $T_1(\delta)$ and $N_1(T_1(\delta))$ that for $t > T_1$ and $c > N_1$, $\left|S_{\Phi_t}^c(u_0)\right| < \delta$; 3) from the existence of $S_F^\infty(u_0) = 0$ follows the existence of $S_{\bar{F}}^\infty(u_0) = 0$, where $\bar{F}(x) \equiv F(x + x_0) = c$ is the family obtained from $F(x) = c$ by a shift by an arbitrary but fixed vector $-x_0$; 4) by virtue of (1) and (2), for any c, Φ, t the inequalities

$$k_4 r^n(c, \varphi, t) \leq \operatorname{mes} V_{\Phi_t}^c \leq k_5 r^n(c, \varphi, t).$$

hold. In view of the remarks made, it is sufficient to study

$$u(t, 0) = \int G(t, x) u_0(x)\, dx.$$

We shall regard this integral as the limit of integrals over $V_{\Phi_t}^c$ as $c \to \infty$. Accordingly, introduce new variables $c, \varphi_1, \dots, \varphi_{n-1}$:

$$x_1 = r(c, \varphi, t) \cos\varphi_1 \cdots \cos\varphi_{n-1},$$

$$x_1 = r(c, \varphi, t) \cos\varphi_1 \cdots \cos\varphi_{n-i} \sin\varphi_{n-i+1},$$

$$x_n = r(c, \varphi, t) \sin\varphi_1, \tag{5}$$

$i = 2, 3, \dots, n-1$. Denote by $G_1(t, c) = G(t, x(c, \varphi, t))$, j the functional determinant of transformation (5), $r(c, t) = r(c, 0, t)$, and Ω_t^c the surface $\Phi_t(x) = c$.

In the new variables the estimate (J_1^+), by virtue of (2) and assumption 2), takes the form

$$\left|\frac{\partial^m G_1(t, c)}{\partial c^m}\right| \leq k_2 a(t)^{-m-n} \exp\{-k_6 (r(c, t) a^{-1}(t))^q\} \left(r_c'(c, t)\right)^m, \qquad m = 0, 1.$$

We shall prove that for every $\varepsilon > 0$ there is a T such that, for $t > T$,

$$|u(t, 0)| < \varepsilon.$$

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In the new variables

$$u(t,0) = \int_0^\infty dc \int_{\Omega_t^c} G_1(t,c) u_0(c;\varphi;t)\, d\Omega =$$

$$= \int_0^\infty dc\, G_1(t,c) \frac{\partial}{\partial c} \int_0^c d\sigma \int_{\Omega_t^\sigma} u_0(\sigma,\varphi,t)\, d\Omega.$$

After integration by parts, taking into account (J_1^+), (2), and remark 4), we obtain

$$u(t,0) \le \int_0^\infty \frac{k_2 r_c'(c,t)\, dc}{a(t)^{n+1} \exp\{k_6 (r(c,t) a^{-1}(t))^q\}} \left| \int_0^c d\sigma \int_{\Omega_t^\sigma} u_0(\sigma,\varphi,t)\, d\Omega \right|.$$

By virtue of remark (2) one can find $N_1(\varepsilon/2k_7)$ and $T_1(\varepsilon/2k_7)$ (k_7 is defined below) such that, for $c > N_1$ and $t > T_1$,

$$|S^c_{\Phi_t}(u_0)| < \varepsilon/2k_7.$$

Denoting by

$$N_2 = \sup_t r(N_1, t)$$

for $t > T_1$, we obtain

$$|u(t,0)| < \int_0^{N_2} \frac{k_8\, dr}{a(t) \exp\{k_9 (r a^{-1}(t))^q\}} + \frac{\varepsilon}{2k_7} \int_{N_2}^\infty \frac{k_8\, dr}{a(t) \exp\{k_9 (r a^{-1}(t))^q\}} <$$

$$< k_8 \frac{N_2}{a(t)} + \frac{\varepsilon}{2k_7} \int_0^\infty \frac{k_8\, dx}{\exp\{k_9 x^q\}}.$$

For t greater than some T_2, $k_8 N_2 a^{-1}(t) < \varepsilon/2$, and for $t > T = \max\{T_1, T_2\}$ we have

$$|u(t,0)| < \varepsilon.$$

3. We shall consider families of hypersurfaces $\overline{\Phi}_t(x) = c$ possessing properties 1°-4°. Along with this, consider any family $F(x) = c$ possessing the same properties.

Theorem 2. If: 1) conditions 1), 3) of Theorem 1 are satisfied; 2) the Green matrix of system (3) is constant in x on each hypersurface $\overline{\Phi}_t(x) = c$; 3)

$$S^\infty_F(u_0) = l;$$

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4) $u_0(x) \geq l$, i.e. the components of the vectors $u_0^i(x) \geq l^i$, then

$$\lim u(t,x) = l$$

uniformly in every finite domain E_n.

4. Since the Green function of the parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t) \frac{\partial u}{\partial x_i} + c(t)u \tag{6}$$

is written explicitly, Theorem 1 admits the following concretization.

Denote

$$A_{ij}(t) = \int_0^t a_{ij}(\tau)\, d\tau, \qquad B_i(t) = \int_0^t b_i(\tau)\, d\tau, \qquad C(t) = \int_0^t c(\tau)\, d\tau,$$

$\overline{A}_{ij}(t)$–the elements of the matrix inverse to $(A_{i,j}(t))_{i,j=1}^n$.

Theorem 3. *Suppose:* 1) $\lim\limits_{t\to\infty} \bar{A}_{ij}(t)[\det A(t)]^{1/n} = \alpha_{ij}$ $(\alpha_{ij} \neq 0)$;

2) $\lim\limits_{t\to\infty} \det A(t) = \infty$; 3) $\lim\limits_{t\to\infty} \dfrac{B_i(t)}{[\det A(t)]^{1/2n}} = \beta_i$; 4) $\lim\limits_{t\to\infty} C(t) = \gamma$; 5) $S_{\bar{F}}^{\infty}(u_0) = l$,

where

$$\bar{F}^2(x) \equiv \sum_{i,j=1}^{n} \alpha_{ij}(x_i - \beta_i)(x_j - \beta_j) = c^2.$$

Then $\lim\limits_{t\to\infty} u(t,x) = le^{\gamma}$ uniformly in every finite region of space.

Theorem 2 in the case of equation (6) turns into Theorem 1 ([2]); if $F(x) = c$– a family of spheres with center at the origin, Theorem 2 ([2]) is a special case of Theorem 3, corresponding to the assumption that the limiting family of surfaces is the same spheres.

5. Theorems 1 and 3 guaranteed stabilization of the solution of the Cauchy problem under assumptions on the existence of a limiting mean over bodies whose character was dictated by the level surfaces of the Green matrix. We give an example showing the naturalness of such an approach. The equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$

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has a Green function whose level lines are the ellipses

$$x_1^2 + \frac{x_2^2}{4} = c^2.$$

Let $u_0(x_1, x_2)$ be a function defined as follows: if by Ω_{mk} we denote the region of the plane determined in polar coordinates by the inequalities

$$m^m \le r < (m+1)^{m+1}, \qquad (2k-1)\pi/4 \le \varphi < (2k+1)\pi/4,$$

then

$$u_0(x_1, x_2) = \begin{cases} 1 & \text{in } \Omega_{mk} \text{ with } m+k \text{ even}, \\ -1 & \text{in } \Omega_{mk} \text{ with } m+k \text{ odd}, \end{cases}$$

$$(k, m = 0, 1, 2, \ldots).$$

Then $u_0(x_1, x_2)$ has a limiting mean over the circles $x_1^2 + x_2^2 = c^2$, but has no limiting mean over the ellipses $x_1^2 + x_2^2/4 = c^2$; the solution of the Cauchy problem constructed from it, as is easy to prove, has no single limit as $t \to \infty$.

In conclusion I express my sincere gratitude to S. D. Eidelman for posing the problems solved here and for valuable guidance in the course of carrying out this work, and also to Yu. N. Valitskii for a useful discussion of the example given.

Chernivtsi State
University

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