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Abstract

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ERGODIC PROPERTIES OF GEODESIC FLOWS ON CLOSED RIEMANNIAN MANIFOLDS OF NEGATIVE CURVATURE

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1. (U)-systems

In this article, as in the preceding one (^{1}), use is made chiefly of the fact that the geodesic flow on a closed Riemannian manifold of negative curvature satisfies certain conditions (U), which are formulated below. It therefore seems appropriate to consider arbitrary dynamical systems satisfying these conditions. Speaking of a dynamical system, throughout this article I shall mean that it is defined on an m-dimensional connected closed smooth manifold W^m, belongs to the class C^{2*} and has an integral invariant. A dynamical system may be a system with continuous or with discrete time. Since a dynamical system with continuous time is often called a flow, I shall call a dynamical system with discrete time a cascade. A flow is defined by specifying on W^m a vector field $f(w)$ of class C^2; by $T^t w$ (t is any real number) is denoted the solution of the system of differential equations

$$\frac{d(T^t w)}{dt} = f(T^t w), \qquad T^0 w = w.$$

A cascade is generated by a C^2-diffeomorphism $T : W^m \to W^m$; T^t (t is any integer) is defined in the usual way by means of iterations.

The tangent space to W^m at the point w will be denoted by R^m_w. The diffeomorphisms T^t induce the corresponding mappings $\widetilde{T}^t$ of tangent spaces:

$$\widetilde{T}^t : R^m_w \to R^m_{T^t w}.$$

(In the case of a flow these are the solutions of the system of equations in variations.)

For a flow the conditions (U) are as follows:

(U1). $f(w) \neq 0$ for all w.

(U2). Every R^m_w decomposes into a direct sum

$$R^m_w = X^k_w \oplus Y^l_w \oplus R^1_w, \qquad \dim X = k \neq 0, \qquad \dim Y = l \neq 0, \tag{1}$$

where R^1_w is generated by the vector $f(w)$, and for $\xi \in X^k_w$, $\eta \in Y^l_w$ we have

$$|\widetilde{T}^t \xi| \leq a|\xi|e^{-ct} \quad \text{for } t \geq 0; \qquad |\widetilde{T}^t \xi| \geq b|\xi|e^{-ct} \quad \text{for } t \leq 0,$$

$$|\widetilde{T}^t \eta| \leq a|\eta|e^{ct} \quad \text{for } t \leq 0; \qquad |\widetilde{T}^t \eta| \geq b|\eta|e^{ct} \quad \text{for } t \geq 0.$$

The constants a, b, c are positive and the same for all w and all ξ, η.

For a cascade the conditions (U) are modified as follows: (U1) is omitted, and in (U2), instead of (1), one speaks of a decomposition $R^m_w = X^k_w \oplus Y^l_w$ with analogous properties. A system (flow, cascade) satisfying conditions (U) I shall call, for short, a (U)-system ((U)-flow, (U)-cascade).

It is easy to show that the subspaces X^k_w and Y^l_w are determined uniquely, that k, l are the same for all w, that

$$\widetilde{T}^t X^k_w = X^k_{T^t w}, \qquad \widetilde{T}^t Y^l_w = Y^l_{T^t w}$$

and that X^k_w, Y^l_w depend continuously on w; however, this dependence is not always smooth (even if the system is analytic).

* The membership of the system in the class C^2 is used only in the proof of Theorem 1; in the remaining arguments only membership in the class C^1 is used directly. In the proof of Theorem 1 an integral invariant is not used.

2. Examples.

A. Under a small (in the sense of C^1) perturbation of a (Y)-system one again obtains a (Y)-system (regardless of whether the unperturbed and perturbed systems have an integral invariant). In essence, this was proved in (^{2}), although there the discussion concerns a special case–small perturbations of certain automorphisms of the torus, mainly of the two-dimensional torus. In this last case $k = l = 1$, so that the fields of tangent spaces X^1_w, Y^1_w are simply fields of directions; in (^{2}) it is proved that these fields belong to the class C^1. The proof is essentially valid for any (Y)-system for which $k = 1$ or $l = 1$. In (^{2}) it was also asserted that the first derivatives of the angular coefficients defining the fields of directions X^1_w, Y^1_w have bounded variation, but the proof that was intended turned out to be erroneous (^{13}), and I do not know whether this assertion is true. One can construct an example in which the fields X^1_w, Y^1_w do not belong to the class C^2.

B. Let a diffeomorphism $T_0 : W_0^m \to W_0^m$ generate a (Y)-cascade. Identifying in the direct product $W_0^m \times [0,1]$ the points $(w,0)$ and $(T_0 w, 1)$, we obtain a new closed manifold W^{m+1}. Define a (Y)-flow $T^t : W^{m+1} \to W^{m+1}$ by putting

$$T^t(w,s) = (T_0^{[t+s]}w,\ t+s-[t+s])$$

(the square brackets denote the integer part). Of course, the study of this flow is entirely reduced to the study of the cascade T_0^n. It is obvious that the spectrum of the flow T^t contains a discrete component.

C. Let V^n be a closed n-dimensional Riemannian manifold of negative curvature (the curvature must be negative at every point and in every two-dimensional direction), and let W^{2n-1} be the space of unit tangent vectors of V^n. The geodesic lines of the manifold V^n determine a "geodesic flow" in W^{2n-1}, which satisfies the conditions (Y)—the latter was in fact proved by Hadamard and É. Cartan (see Appendix III to the book (^{3})). For the geodesic flow $k = l = n-1$; therefore in the case $n = 2$ the fields of tangent subspaces X_w^1, Y_w^1 are smooth (as was known to E. Hopf, see (^{4}), § 14). These fields will also be smooth in the case when the curvature is constant (which Lobachevsky himself already knew) or "almost constant." In the general case, however, they are apparently not smooth.

3. Foliations.

Following (^{5}), I shall say that a smooth p-dimensional manifold M^p is a submanifold of the manifold W^m if the points of M^p are contained among the points of W^m and if the inclusion $M^p \subset W^m$ is a regular mapping. We shall call a submanifold **complete** if it is complete in the Riemannian metric induced on M^p by the Riemannian metric on all of W^m.

A **foliation*** $\mathfrak{S}^p$ is a decomposition of the manifold W^m into connected smooth complete p-dimensional manifolds (called leaves), having the property that if to each point $w \in W^m$ one assigns the p-dimensional vector subspace of the tangent space tangent at that point to the leaf passing through it, then the resulting field of p-dimensional spaces (it is called the tangent field of the foliation $\mathfrak{S}^p$) is continuous. A foliation is called **smooth** if its tangent field is smooth.

Take some small $(n-p)$-dimensional plaque Π, transversal to the leaves of $\mathfrak{S}^p$. Suppose that near Π there is another such plaque Π', and that every leaf passing through some point w of the plaque Π intersects Π' at a point w', which is close to w in the metric of the leaf. Consider the mapping $\varphi : \Pi \to \Pi'$, taking w to w'.

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* I use this term as a synonym of the terms "layered structure," "leafage," corresponding to the French *feuilletage* and the English "foliated manifold." In speaking of a foliation, I always mean that the leaves are smooth.

If for every such pair of nearby plaques the corresponding mapping φ has a continuous generalized Jacobian, and, under a small deformation of the plaque Π', this Jacobian changes continuously, then I shall call the foliation $\mathfrak{S}^p$ **absolutely continuous**.

Finally, a foliation is called **metrically transitive** if, for any measurable set $A \subset W^m$ consisting entirely of leaves, either $\operatorname{mes} A = 0$, or $\operatorname{mes} W^m \setminus A = 0$.

It turns out that the fields X_w^k, Y_w^l are tangent fields of certain foliations, which I shall denote by $\mathfrak{S}^k$ and $\mathfrak{S}^l$, respectively, and which are invariant with respect to T^t in the sense that a leaf is carried into (generally speaking) another leaf.

4. Results.

Theorem 1. *The foliations $\mathfrak{S}^k, \mathfrak{S}^l$ are absolutely continuous.*

This theorem is new. It is used in the proof of Theorems 2–5*.

Theorem 2. *Every (Y)-system is ergodic.*

Simultaneously with the author, Theorem 2 was proved by Ya. G. Sinai. In example B, ergodicity had previously been established for a number of special cases ([4], [6]); see also ([7]) and the remark in the abstract ([8]) (in contrast to these authors, I did not consider open manifolds). From Theorem 2 it is easy to deduce that the spectrum of any (Y)-cascade is continuous. With spectra of (Y)-flows, as is seen from example B, the matter is more complicated.

Theorem 3. *If a (Y)-flow $T^t : W^m \to W^m$ has an eigenfunction different from a constant, then this function is continuous and there exist a closed submanifold $W_0^{m-1} \subset W^m$ and a (Y)-cascade $T_0^n : W_0^{m-1} \to W_0^{m-1}$ such that the flow T^t is obtained from the cascade T_0^n by means of the construction described in example B (up to a change of the time scale).*

For geodesic flows (example B), the latter possibility drops out ([9], [10]).

Theorem 4. *For any (Y)-cascade the foliations $\mathfrak{S}^k, \mathfrak{S}^l$ are metrically transitive.*

Theorem 5. *If a (Y)-flow has continuous spectrum, then the foliations $\mathfrak{S}^k, \mathfrak{S}^l$ are metrically transitive.*

For the geodesic flow (example B) in the two-dimensional case this was proved somewhat earlier by another method by Ya. G. Sinai; he also proved Theorem 4. The theory he constructed makes it possible to deduce from the metric transitivity of the foliations $\mathfrak{S}^k$ and $\mathfrak{S}^l$ that the dynamical system has mixing of all degrees, countable Lebesgue spectrum, positive entropy, and, in general, is a K-system ([11], [12]).

Remark added in proof. There exist perturbations of an ergodic automorphism of the torus under which the Euclidean measure remains invariant, but the entropy changes.

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* What has been said applies both to my proofs and to the proofs mentioned below by Ya. G. Sinai, who assumed (and did not prove) the absolute continuity of the foliations.

Note: Figure translations are in progress. See original paper for figures.

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