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Abstract

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ON THE INDEX AND NORMAL SOLVABILITY OF ONE CLASS OF FUNCTIONAL EQUATIONS

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On a simple closed Lyapunov contour L there is given a function $\alpha(t)$, having a derivative $\alpha'(t)$, satisfying on L a Hölder condition ($\alpha'(t) \subset H$), and moreover $\alpha'(t) \neq 0$. Let the function $\alpha(t)$ effect a homeomorphic mapping of the contour L onto itself (with preservation or change of orientation on L) and satisfy T. Carleman' s condition (^{1})

$$\alpha[\alpha(t)] \equiv t. \tag{1}$$

Consider the functional equation

$$\begin{aligned} K_\mu\varphi \equiv a(t)\varphi(t) + \mu b(t)\varphi[\alpha(t)] + \frac{c(t)}{\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}\,d\tau + \mu\frac{d(t)}{\pi i}\int_L \frac{\varphi(\tau)}{\tau - \alpha(t)}\,d\tau \\ + \int_L m_1(t,\tau)\varphi(\tau)\,d\tau + \mu\int_L m_2[\alpha(t),\tau]\varphi(\tau)\,d\tau = h(t), \qquad \mu = \pm 1. \end{aligned} \tag{2}$$

Here $a(t), b(t), c(t), d(t), h(t) \subset H$ on L; $m_1(t,\tau), m_2(t,\tau)$ are Fredholm kernels.

The functional equation (2), under certain additional conditions imposed on its coefficients, admits a complete investigation, carried out earlier by the author (^{2,3}). In the present article a new approach is proposed to the study of equation (2), based on reducing this equation to a system of singular integral equations with Cauchy kernels. This method makes it possible to find exact limits for the applicability of Noether theory to equation (2).

§ 1. Consider the system of singular equations

$$\begin{aligned} a(t)\rho_1(t) + \nu b(t)\rho_2(t) + \frac{c(t)}{\pi i}\int_L \frac{\rho_1(\tau)}{\tau - t}\,d\tau + \frac{\nu\lambda d(t)}{\pi i}\int_L \frac{\alpha'(\tau)\rho_2(\tau)}{\alpha(\tau) - \alpha(t)}\,d\tau \\ + \int_L m_1(t,\tau)\rho_1(\tau)\,d\tau + \nu\lambda\int_L m_2[\alpha(t),\alpha(\tau)]\alpha'(\tau)\rho_2(\tau)\,d\tau = h(t), \end{aligned} \tag{3}$$

$$\begin{aligned} & b[\alpha(t)]\rho_1(t) + \nu a[\alpha(t)]\rho_2(t) + \frac{d[\alpha(t)]}{\pi i}\int_L \frac{\rho_1(\tau)}{\tau - t}\,d\varepsilon \\ & \quad + \frac{\nu\lambda c[\alpha(t)]}{\pi i}\int_L \frac{\alpha'(\tau)\rho_2(\varepsilon)}{\alpha(\tau)-\alpha(t)}\,d\tau + \int_L m_2(t,\tau)\rho_1(\tau)\,d\tau \\ & \quad + \nu\lambda\int_L m_1'[\alpha(t),\alpha(\tau)]\alpha'(\tau)\rho_2(\tau)\,d\varepsilon = \nu h[\alpha(t)], \qquad \nu = \pm 1. \end{aligned}$$

Here $\lambda = 1$ if $\alpha(t)$ preserves the orientation on L, and $\lambda = -1$ if the orientation on L changes.

Let l be the number of linearly independent solutions of the homogeneous system of equations (3), and let k_+ and k_- be the numbers of linearly independent solutions of the homogeneous equation (2), respectively for $\mu = 1$ and $\mu = -1$. The following assertions are valid: a) $l = k_+ + k_-$; b) there exist l linearly indepen-

independent solutions of the homogeneous system (3) such that k_+ solutions satisfy the condition $\rho_2[\alpha(t)] - \nu\rho_1(t) = 0$, and the remaining k_- solutions satisfy the condition $\rho_2[\alpha(t)] + \nu\rho_1(t) = 0$; c) the nonhomogeneous equation (2) is solvable if and only if the nonhomogeneous system (3) is solvable: if equation (2) is solvable and $\varphi(t)$ is its solution, then the functions $\rho_1(t) = \varphi(t)$, $\rho_2(t) = \mu\nu\varphi[\alpha(t)]$ are a solution of system (3); conversely, if system (3) is solvable and $\{\rho_1(t), \rho_2(t)\}$ is a solution of this system, then the function $\varphi(t) = \frac{1}{2}\{\rho_1(t) + \mu\nu\rho_2[\alpha(t)]\}$ is a solution of equation (2).

Let us also consider the functional equation

$$\begin{aligned} K_\mu'\psi \equiv a(t)\psi(t) + \mu\lambda\alpha'(t)b[\alpha(t)]\psi[\alpha(t)] - \frac{1}{\pi i}\int_L \frac{c(\tau)\psi(\tau)}{\tau - t}\,d\tau & \\ - \frac{\mu\lambda}{\pi i}\int_L \frac{d[\alpha(\tau)]\alpha'(\tau)\psi[\alpha(\tau)]}{\tau - t}\,d\tau + \int_L m_1(\tau,t)\psi(\tau)\,d\tau & \qquad (4) \\ + \mu\lambda\int_L m_2(\tau,t)\alpha'(\tau)\psi[\alpha(\tau)]\,d\tau = 0, & \end{aligned}$$

adjoint to equation (2). Let k_+' and k_-' be the numbers of linearly independent solutions of equation (4), respectively for $\mu = 1$ and $\mu = -1$, and let l' be the number of linearly independent solutions of the system

$$a(t)\omega_1(t) + \nu b[\alpha(t)]\omega_2(t) - \frac{1}{\pi i}\int_L \frac{c(\tau)\omega_1(\tau)}{\tau - t}\,d\tau - \frac{\nu}{\pi i}\int_L \frac{d[\alpha(\tau)]\omega_2(\tau)}{\tau - t}\,d\tau + \int_L m_1(\tau,t)\omega_1(\tau)\,d\tau + \nu\int_L m_2(\tau,t)\omega_2(\tau)\,d\tau \qquad (5)$$

$$b(t)\omega_1(t) + \nu a[\alpha(t)]\omega_2(t) - \frac{\lambda\alpha'(t)}{\pi i}\int_L \frac{d(\tau)\omega_1(\tau)}{\alpha(\tau)-\alpha(t)}\,d\tau$$
$$- \frac{\nu\lambda\alpha'(t)}{\pi i}\int_L \frac{c[\alpha(\tau)]\omega_2(\tau)}{\alpha(\tau)-\alpha(t)}\,d\tau + \lambda\alpha'(t)\int_L m_2[\alpha(\tau),\alpha(t)]\omega_1(\tau)\,d\tau$$
$$+ \nu\lambda\alpha'(t)\int_L m_1[\alpha(\tau),\alpha(t)]\omega_2(\tau)\,d\tau = 0,$$

adjoint to system (3). We have $l' = k'_+ + k'_-$, where k'_+ and k'_- solutions satisfy, respectively, the conditions

$$\omega_2(t) - \nu\lambda\alpha'(t)\omega_1[\alpha(t)] = 0, \qquad \omega_2(t) + \nu\lambda\alpha'(t)\omega_1[\alpha(t)] = 0.$$

Denote $A(t) = a(t)+c(t)$, $C(t) = c(t)-a(t)$, $B(t) = b(t)+d(t)$, $D(t) = d(t)-b(t)$.

Using the known condition for a system of singular equations with Cauchy kernels to belong to the normal Noetherian type (^{4}), we obtain the following principal theorem.

Theorem 1. *In order that the equation $K_\mu\varphi = h(t)$ be a Noether equation, it is necessary and sufficient that the following conditions hold on L:*

a) $\theta(t) = B(t)D[\alpha(t)] - A[\alpha(t)]C(t) \neq 0$, if $\lambda = -1$;

b) $\theta_1(t) = C(t)C[\alpha(t)] - D(t)D[\alpha(t)] \neq 0$, $\theta_2(t) = A(t)A[\alpha(t)] - B(t)B[\alpha(t)] \neq 0$, if $\lambda = 1$.

It is easy to formulate Noether theorems for the functional equation (2).

Theorem 2. The index of the functional equation $K_\mu\varphi = 0$ over the field of real numbers is expressed by the formulas:

$$I = 2\operatorname{Ind}\theta(t), \qquad \text{if } \lambda = -1; \tag{6}$$

$$I = \operatorname{Ind}\theta_1(t) - \operatorname{Ind}\theta_2(t), \qquad \text{if } \lambda = 1. \tag{7}$$

§ 2. The numbers of linearly independent solutions of the characteristic ($m_1(t,\tau) = m_2(t,\tau) \equiv 0$) equations $K_\mu\varphi = 0$ and $K'_\mu\psi = 0$ (and these solutions themselves) are completely determined in the following cases:

1) $\lambda = -1,\ \theta(t) \neq 0,\ \theta_1(t) = \theta_2(t) \equiv 0$;

2) $\lambda = 1,\ \theta_1(t) \neq 0,\ \theta_2(t) \neq 0,\ \theta(t) \equiv 0$;

3) $\Lambda(t) = B(t)C[\alpha(t)] - D(t)A[\alpha(t)] \equiv 0$ on L

and one of the conditions a) or b) of Theorem 1 must be satisfied. Indeed, the characteristic equation (2) is equivalent to a system of two Carleman problems (^{3}) in case 1), to a Haseman problem (^{5}) in case 2), and to a Riemann problem in case 3). Let us note* that case 1) was investigated in the author' s paper (^{3}). There the formula (6) was also established for this case.

§ 3. Let us consider several special cases of the functional equation (2).

1. Let $b(t) = c(t) \equiv 0$ on L. If the conditions

$$\theta(t) = a(t)a[\alpha(t)] + d(t)d[\alpha(t)] \neq 0 \qquad \text{for } \lambda = -1,$$

$$\theta_1(t) \equiv \theta_2(t) = a(t)a[\alpha(t)] - d(t)d[\alpha(t)] \neq 0 \qquad \text{for } \lambda = 1,$$

are satisfied, then for equation (2) the Fredholm alternative is valid ($I = 0$) (see also (^{2})).

2. Let $c(t) = b(t),\ d(t) = -a(t)$. Suppose that $\lambda = -1$. Then condition a) of Theorem 1 takes the form $a^2(t) - b^2(t) \neq 0$;

$$I = 2\,\mathrm{Ind}\,\frac{a(t) - b(t)}{a(t) + b(t)}.$$

If $\lambda = 1$, then

$$\theta_1(t) = -\theta_2(t) = -2\{a(t)b[\alpha(t)] + a[\alpha(t)]b(t)\} \neq 0;$$

$$I = 0.$$

Results of an analogous character are obtained if $c(t) = -b(t),\ d(t) = a(t)$. Let also: a) $c(t) = b(t),\ d(t) = a(t)$ or b) $c(t) = -b(t),\ d(t) = -a(t)$. Then for $\lambda = -1$ we have

$$\theta(t) = 2\{a(t)a[\alpha(t)] - b(t)b[\alpha(t)]\} \neq 0$$

and $I = 0$, while for $\lambda = 1$ equation (2) is not a Noether equation, since $\theta_1(t) = \theta_2(t) \equiv 0$.

3. Let $c(t) = -a(t),\ d(t) = b(t)$. If $\lambda = -1$, then $\theta(t) \equiv 0$, and equation (2) does not belong to the Noether type. If $\lambda = 1$, then condition b) of Theorem 1 gives $a(t) \neq 0,\ b(t) \neq 0$, and from Theorem 2 we have that

$$I = 2\,\mathrm{Ind}\,\frac{a(t)}{b(t)}$$

(see (4), p. 227). We have analogous results if $c(t) = a(t)$, $d(t) = -b(t)$. Finally, if: a) $c(t) = a(t), d(t) = -b(t)$ or b) $c(t) = -a(t), d(t) = b(t)$, then the corresponding equation (2) is not a Noether equation either for $\lambda = 1$ or for $\lambda = -1$.

4. Let: 1) $c(t) = d(t) \equiv 0$ or 2) $a(t) = b(t) \equiv 0$. Both functional equations belong to the Fredholm type.

§ 4. The characteristic equation (2) is reduced to the problem of finding a piecewise-analytic function $\{\Phi^+(z), \Phi^-(z)\}$, vanishing at infinity, from the boundary condition on L

$$A(t)\Phi^+(t) + \mu B(t)\Phi^+[\alpha(t)] + C(t)\Phi^-(t) + \mu D(t)\Phi^-[\alpha(t)] = h(t). \tag{8}$$

If in (8) we put $C(t) = D(t) \equiv 0$ or $A(t) = B(t) \equiv 0$, then we arrive at the boundary-value problems on L

$$A(t)\Phi^+(t) + \mu B(t)\Phi^+[\alpha(t)] = h(t), \quad C(t)\Phi^-(t) + \mu D(t)\Phi^-[\alpha(t)] = ht. \tag{9}$$

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* In the paper (^{3}), to the condition $a(t), b(t), c(t), d(t) \neq 0$ on L, one should add the condition: the inequality $\theta(t) \neq 0$ is satisfied on L everywhere except for a finite number of points of L, $g_+(t), g_-(t) \in H$ on L. Of course, all the theorems of paper (^{3}) are valid if the condition $a(t), b(t), c(t), d(t) \neq 0$ on L is replaced by condition a) of the present paper.

The conditions of Theorem 1 for the boundary value problems (9) are not satisfied and, consequently, the problems (9) do not belong to the normal Noetherian type. This result is easy to explain. If $\lambda = 1$, then the problems (9) belong to the type of so-called one-sided problems considered by E. I. Zverovich and the author [6]. As shown in [6], such problems may have an infinite set of linearly independent solutions. If $\lambda = -1$, then we have the well-known problems of T. Carleman. The violation of Noether' s theorems here consists in the fact that fulfillment of the conditions of normal solvability (the second Noether theorem) turns out to be insufficient for solvability of these problems; in addition (for example, for the first problem (9)) L conditions must be satisfied:

$$A(t)A[\alpha(t)] = B(t)B[\alpha(t)], \qquad \mu B[\alpha(t)]h(t) + A(t)h[\alpha(t)] = 0.$$

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Note: Figure translations are in progress. See original paper for figures.

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