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Abstract

Full Text

MATHEMATICS

M. A. NAIMARK

ON COMMUTING UNITARY OPERATORS IN THE SPACE Π_χ

(Presented by Academician L. S. Pontryagin, 17 XI 1962)

A space Π_χ (where χ is a natural number) is a Hilbert space in which, in addition to the usual scalar product $[x, y]$, a second indefinite scalar product (x, y) is given, defined, for a suitable choice of an orthonormal basis $\{e_\alpha\}$, by the formula

$$(x, y) = \sum_{\alpha=1}^{\chi} [x, e_\alpha]\overline{[y, e_\alpha]} - \sum_{\alpha>\chi} [x, e_\alpha]\overline{[y, e_\alpha]}.$$

An axiomatic definition of the space Π_χ was given in (1); here we retain the terminology of article (1). A linear operator U mapping Π_χ one-to-one onto itself is called **unitary** if $(Ux, Uy) = (x, y)$ for all $x, y \in \Pi_\chi$. According to a theorem of L. S. Pontryagin (2), for every unitary operator U in Π_χ there exists a χ-dimensional nonnegative subspace invariant with respect to U^*.* For $\chi = 1$ the corresponding invariant subspace is one-dimensional and, consequently, determines a nonnegative eigenvector of the unitary operator.

The following theorem will be useful in constructing the theory of unitary representations of groups in the spaces Π_χ and in a number of other questions.

Theorem 1. *Let $\mathfrak{U}$ be a set of pairwise commuting unitary operators in the space Π_χ. Then in Π_χ there exists a nonnegative χ-dimensional subspace invariant with respect to all operators $U \in \mathfrak{U}$.*

For simplicity and brevity of exposition, we shall give the proof here only for $\chi = 1$.

1. Choose an orthonormal basis $\{e_\alpha\}$ in Π_1 so that

$$(x, y) = x_1\overline{y_1} - \sum_{\alpha>1} x_\alpha\overline{y_\alpha}, \tag{1}$$

where $x_\alpha = [x, e_\alpha]$, $y_\alpha = [y, e_\alpha]$. If the vector x is nonnegative, then, by virtue of (1),

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$$0 \leq (x,x) = |x_1|^2 - \sum_{\alpha>1} |x_\alpha|^2; \tag{2}$$

hence it follows that $x_1 \neq 0$ for $x \neq 0$. Replacing x by $\frac{1}{x_1}x$, we may assume that $x_1 = 1$. In this case, from (2) we conclude that

$$\sum_{\alpha>1} |x_\alpha|^2 \leq 1. \tag{3}$$

A nonnegative vector x will be called **normalized** if $x_1 = [x, e_1] = 1$.

2. Let K be the totality of all normalized nonnegative vectors in Π_1; by (3) there exists a one-to-one correspondence between the points of K and the points of the unit ball Q in the (ordinary) Hilbert space l^2 of all numerical sequences $\{x_\alpha, \alpha > 1\}$ satisfying the condition $\sum_{\alpha>1} |x_\alpha|^2 < \infty$. With the aid of this correspondence we transfer to K the weak topology in Q; then K becomes a bicompact space, since Q is bicompact in the weak topology.

3. Let $U \in \mathfrak{U}$. If $x \in K$, then $x \neq 0$, $(x,x) \geqslant 0$; putting $y = Ux$, we have:

$$y \neq 0, \qquad (y,y) = (Ux,Ux) = (x,x) \geqslant 0,$$

therefore $y_1 = (y, e_1) \neq 0$. Introduce the (nonlinear) operator $V(x)$, putting $V(x) = y_1^{-1}y = (Ux, e_1)^{-1}Ux$. The operator V maps K into itself. Indeed, if $x \in K$, then $(x,x) \geqslant 0$, hence also

$$(V(x), V(x)) = |y_1|^{-2}(Ux, Ux) = |y_1|^{-2}(x,x) \geqslant 0;$$

moreover, $(V(x), e_1) = y_1^{-1}(Ux, e_1) = y_1^{-1}y_1 = 1$. It is also obvious that the operator V is continuous on K in the weak topology. On the basis of Schauder' s theorem we conclude from this that K contains a fixed point x of the operator V, i.e. $V(x) = x$*. This means that $y_1^{-1}Ux = x$, i.e. $Ux = y_1 x$; thus the existence of a nonnegative eigenvector has been proved, and hence of a nonnegative one-dimensional invariant subspace of the operator U**.

4. From the preceding reasoning it is clear that to each fixed point x of the operator V in K there corresponds a one-dimensional nonnegative invariant

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* L. S. Pontryagin established this theorem for self-adjoint (with respect to (x,y)) operators; by virtue of the Cayley transform, the result of L. S. Pontryagin is equivalent to the assertion about unitary operators formulated in the text. Another, simpler proof of this theorem for unitary (and more general non-decreasing) operators was subsequently given in the article of M. G. Krein (^{3}) (see also (^{1})).

subspace of the operator U. Conversely, let $\mathfrak{M}_1$ be such a subspace, and let x be a normalized vector from $\mathfrak{M}_1$. Then $Ux = \lambda x$ for some λ. Hence $y_1 = (Ux, e_1) = \lambda(x, e_1) = \lambda x_1 = \lambda$; consequently, $x = \lambda^{-1}Ux = y_1^{-1}Ux = V(x)$, i.e. x is a fixed point of the operator V in K.

5. Let $F(U)$ be the set of all fixed points of the operator $V(x) = y_1^{-1}Ux$ in K; by the continuity of the operator V in K, the set $F(U)$ is a closed subset in K. From the arguments in §§ 3-4 we conclude that the assertion of the theorem will be proved if we establish that the intersection of all $F(U)$, $U \in \mathfrak{U}$, is nonempty. On the basis of the bicompactness of the set K, for this it is in turn sufficient to show that the intersection of any finite number of the sets $F(U)$ is nonempty. In view of what was said in §§ 3-4, for this it is sufficient to establish that for any finite number of operators $U_1, \ldots, U_n \in \mathfrak{U}$ there exists a nonnegative one-dimensional subspace invariant with respect to all U_j, $j = 1, \ldots, n$.

We shall prove this last assertion by induction on n. For $n = 1$ the assertion coincides with what was proved in § 3. Suppose that the assertion has been proved for some n; we shall prove that it is valid for $n + 1$.

Let $U_1, \ldots, U_n, U_{n+1} \in \mathfrak{U}$. By the induction hypothesis, in Π_1 there exists a one-dimensional nonnegative subspace $\mathfrak{M}_1$ invariant with respect to all U_j, $j = 1, \ldots, n$; consequently, for $x \in \mathfrak{M}_1$

$$U_{jx} = \lambda_{jx} \qquad \text{for all } j = 1, \ldots, n, \tag{4}$$

where λ_j are certain numbers. Denote by $\mathfrak{M}$ the totality of all vectors $x \in \Pi_1$ satisfying condition (4); then $\mathfrak{M}$ is a subspace in Π_1, and $\mathfrak{M}_1 \subset \mathfrak{M}$. If $x \in \mathfrak{M}$, so that (4) holds, then $U_{jU_{n+1}}x = U_{n+1}U_{jx} = \lambda_{jU_{n+1}}x$; consequently, also $U_{n+1}x \in \mathfrak{M}$. Therefore $U_{n+1}\mathfrak{M} \subset \mathfrak{M}$. Applying the same reasoning to U_{n+1}^{-1}, we conclude that also $U_{n+1}^{-1}\mathfrak{M} \subset \mathfrak{M}$, and therefore

$$U_{n+1}\mathfrak{M} = \mathfrak{M}. \tag{5}$$

* Indeed, by virtue of the correspondence between K and Q established in § 2, V may be regarded as a weakly continuous mapping of the convex bicompact set Q into itself.

** The reasoning in § 3 is a slight modification of the proof of Theorem 3.1 in (^{1}) for $\varkappa = 1$.

Only the following cases are possible:

I. $\mathfrak{M}$ is one-dimensional; hence $\mathfrak{M} = \mathfrak{M}_1$. Then, by virtue of (4) and (5), $\mathfrak{M} = \mathfrak{M}_1$ is a nonnegative one-dimensional subspace invariant with respect to $U_1, \ldots, U_n, U_{n+1}$.

II. $\mathfrak{M}$ is not one-dimensional; by Lemma 1.2, from (1) $\mathfrak{M}$ cannot be nonnegative. In this case only the following cases are possible:

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IIa. The scalar product degenerates on $\mathfrak{M}$. Let $\mathfrak{M}'$ be an isotropic subspace for $\mathfrak{M}$; on the basis of Lemma 1.2, from (1) $\mathfrak{M}'$ is one-dimensional. $\mathfrak{M}'$ is invariant with respect to U_{n+1}. Indeed, if $x \in \mathfrak{M}'$ and $y \in \mathfrak{M}$, then, by virtue of (5), $U_{n+1}^{-1}y \in \mathfrak{M}$, and therefore $0 = (x, U_{n+1}^{-1}y) = (U_{n+1}x, y)$; this means that also $U_{n+1}x \in \mathfrak{M}'$. But then $\mathfrak{M}'$ is a one-dimensional nonnegative subspace invariant with respect to $U_1, \dots, U_n, U_{n+1}$. Case IIa occurs, in particular, if $(x, x) \leqslant 0$ on $\mathfrak{M}$. Indeed, then $(x, x) = 0$ on $\mathfrak{M}_1$, since $\mathfrak{M}_1$ is nonnegative; hence, by the Cauchy—Bunyakovsky inequality $|(x, y)|^2 \leqslant [-(x, x)][-(y, y)]$, valid in the case under consideration for all $x, y \in \mathfrak{M}$, $(x, y) = 0$ for all $y \in \mathfrak{M}$ and $x \in \mathfrak{M}_1$.

IIb. The scalar product does not degenerate on $\mathfrak{M}$. By what was said above, in this case (x, x) changes sign on $\mathfrak{M}$; therefore $\mathfrak{M}$ is also a space Π_1, and U_{n+1} is a unitary operator in $\mathfrak{M}$. By what was proved in item 3, there exists in $\mathfrak{M}$ a one-dimensional nonnegative subspace invariant with respect to U_{n+1}; evidently, it is also invariant with respect to $U_1, \dots, U_n$.

Thus, in every case there exists a nonnegative one-dimensional subspace invariant with respect to $U_1, \dots, U_n, U_{n+1}$; this completes the proof of the theorem for $\kappa = 1$.

Corollary 1. *For every family of mutually commuting bounded self-adjoint operators in the space Π_κ, there exists a κ-dimensional nonnegative subspace invariant with respect to all operators of this family.*

Corollary 2. *Let R be a commutative ring of bounded linear operators in the space Π_κ, containing together with each operator A the operator A^* adjoint (with respect to (x, y)). Then in Π_κ there exists a nonnegative κ-dimensional subspace invariant with respect to all operators from R.*

The following Theorem 2 generalizes Corollary 1 to some noncommutative families of operators; assertion 2) in Theorem 2 may be regarded as an infinite-dimensional analogue of Li’ s theorem.

Theorem 2. *Let X be a set of linear bounded operators in Π_κ possessing the following properties: 1) X is linear with respect to the operations of addition of operators and multiplication of an operator by a number; 2) in X there exists a family of subspaces $X_1, X_2, \dots, X_m$ and Hermitian operators $H_k \in X_k$ such that: a) $X \supset X_1 \supset \cdots \supset X_m$; b) X_k is the subspace generated by the operator H_k and the subspace X_{k+1}, for $k = 0, 1, \dots, m-1$, where, by definition, $X_0 = X$; c) $H_{kH} - HH_k \in X_{k+1}$ for any $H \in X_{k+1}$, $k = 0, 1, \dots, m-1$; d) X_m is commutative and, together with each operator A, contains the operator A^* adjoint (with respect to (x, y)). Then: 1) in Π_κ there exists a nonnegative κ-dimensional subspace invariant with respect to all operators from X; 2) in Π_κ there exists a nonnegative vector $x_0 \neq 0$ which is a common eigenvector for all operators from X.*

We note that in the case of an ordinary Hilbert space, a proposition analogous to assertion 2) generally does not hold.

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