A. Kh. GUDIEV

1963

SovietRxiv

Abstract

Full Text

A. Kh. GUDIEV

THE PROBLEM OF S. L. SOBOLEV AND S. M. NIKOLSKII FOR THE LIMITING EXPONENT

(Presented by Academician S. L. Sobolev on 6 X 1962)

Denote by $\mathbf{L}_{(p_1,p_2)}(R^n)$ the class of functions $f(x)$, defined on R^n, for which the norm is bounded

$$\|f\|_{\mathbf{L}_{(p_1,p_2)}(R^n)} = \left[\int_{R_x^{n-s}} \left(\int_{R_x^s} |f(\bar{x})|^{p_1}\, d\bar{x}_s\right)^{p_2/p_1} d\bar{x}_{n-s}\right]^{1/p_2}.$$

The aim of the present note is to prove the following theorem:

Theorem 1. If $f(\bar{x}) \in W_p^{(l)}(\Omega)$,

$$\frac{n}{p} - l = \frac{n-s}{p_2} + \frac{s}{p_1}, \qquad p_2 \geq p_1 > p > 1,$$

then $f(\bar{x}) \in \mathbf{L}_{(p_1,p_2)}(\Omega)$ and, moreover, the inequality

$$\|f\|_{\mathbf{L}_{(p_1,p_2)}(\Omega)} \leq c\|f\|_{W_p^{(l)}(\Omega)}$$

holds, where c is a constant independent of f (Ω may also be unbounded).

This theorem gives a solution of the problem of S. L. Sobolev and S. M. Nikolskii for the limiting exponent, posed by them at the Fourth All-Union Mathematical Congress.

For the proof of Theorem 1 it suffices to establish the validity of the following theorem, which, in turn, generalizes some results obtained by S. L. Sobolev (^{1}), V. I. Kondrashev (^{2}), V. P. Il' in (^{3}), and L. V. Kantorovich (^{4}).

Theorem 2. If $f(\bar{y}) \in \mathbf{L}_p(R^n)$,

$$\lambda = \frac{n}{p'} + \frac{n-s}{p_2} + \frac{s}{p_1}, \qquad p_2 \geq p_1 > p > 1,$$

then

$$U(\bar{x}) = \int_{R^n} f(\bar{y}) r^{-\lambda}\, d\bar{y} \in \mathbf{L}_{(p_1,p_2)}(R^n) \tag{1}$$

and, moreover, the inequality

$$\|U\|_{\mathbf{L}_{(p_1,p_2)}(R^n)} \le c\|f\|_{\mathbf{L}p(R^n)} \tag{2}$$

holds, where c is a constant independent of f.

If one uses a known theorem of functional analysis, then the proof of inequality (2) is equivalent to the proof of the inequality

$$I = \int_{R_x^n}\int_{R_y^n} \frac{f(\bar{y})\,\varphi_2(\bar{x})\,\varphi_1^{1/p_1}(\bar{x}_{n-s})}{s^\lambda}\,d\bar{y}\,d\bar{x} \le c\|\varphi_2\|_{\mathbf{L}_{p_1'}(R^n)}\|f\|_{\mathbf{L}p(R^n)}\|\varphi_1\|^{1/p_1}_{\mathbf{L}_{(p_1/p_1)'}(R)^{n-s}}, \tag{3}$$

where

$$(p_2/p_1)^{-1} + [(p_2/p_1)']^{-1} = 1.$$

Introduce the notation:

$$r_1 = \left[\sum_1^s y_i^2\right]^{1/2}; \qquad r_2 = \left[\sum_1^s x_i^2\right]^{1/2}; \qquad r_3 = \left[\sum_1^s (x_i - y_i)^2\right]^{1/2};$$

$$r_4 = \left[\sum_{s+1}^n (x_i - y_i)^2\right]^{1/2}; \qquad \bar{x}_s = (x_1, x_2, \dots, x_s); \qquad \bar{x}_{n-s} = (x_{s+1}, x_{s+2}, \dots, x_n).$$

We transform the left-hand side of inequality (3)

$$I = \int_{R_x^{n-s}} \varphi_1^{1/p_1}(\bar{x}_{n-s}) \left[\int_{R_y^{n-s}} \left(\int_{R_x^s}\int_{R_y^s} \frac{f(\bar{y}_s, \bar{y}_{n-s})\varphi_2(\bar{x}_s, \bar{x}_{n-s})}{r^\lambda}\,d\bar{y}_s\,d\bar{x}_s\right) d\bar{y}_{n-s}\right] d\bar{x}_{n-s}.$$

In estimating this integral we may assume that f, φ_1, φ_2 are nonnegative. On the basis of S. L. Sobolev' s lemma (1) and its generalization (see (3)) we have

$$I \le \int_{R_x^{n-s}} \varphi_1^{1/p_1}(\bar{x}_{n-s}) \left[\int_{R_y^{n-s}} \left(\int_{R_x^s}\int_{R_y^s} \frac{f^*(r_1, \bar{y}_{n-s})\varphi_2^*(r_2, \bar{x}_{n-s})}{r^\lambda}\,d\bar{y}_s\,d\bar{x}_s\right) d\bar{y}_{n-s}\right] d\bar{x}_{n-s}, \tag{4}$$

where $f^*(r_1, \bar{y}_{n-s})$ is a nonincreasing function of r_1 for all $\bar{y}_{n-s} = (y_{s+1}, y_{s+2}, \dots, y_n)$; $\varphi_2^*(r_2, \bar{x}_{n-s})$ is a nonincreasing function of r_2 for all $\bar{x}_{n-s} = (x_{s+1}, x_{s+2}, \dots, x_n)$.

We split the integral in the round brackets on the right-hand side of inequality (4) into three summands B_1, B_2, and B_3, where

$$B_i = \iint_{\substack{r_k \le r_i \\ r_j \le r_i}} \frac{f^*(r_1, \bar{y}_{n-s}) \varphi_2^*(r_2, \bar{x}_{n-s})}{r^\lambda} \, d\bar{x}_s \, d\bar{y}_s \tag{5}$$

$$(i \ne k;\ i \ne j;\ k \ne j;\ k, j, i = 1, 2, 3);$$

then

$$I \le \sum_{i=1}^{3} \int_{R_x^{n-s}} \varphi_1^{1/p_1}(\bar{x}_{n-s}) \left[\int_{R_y^{n-s}} B_i \, d\bar{y}_{n-s} \right] d\bar{x}_{n-s} = \sum_{i=1}^{3} I_i,$$

where

$$I_i = \int_{R_x^{n-s}} \varphi_1^{1/p_1}(\bar{x}_{n-s}) \left[\int_{R_y^{n-s}} B_i \, d\bar{y}_{n-s} \right] d\bar{x}_{n-s}.$$

In view of the fact that the quantities I_1, I_2, and I_3 are estimated analogously, it is sufficient to estimate one of them, for example I_2.

To estimate B_2 we pass to polar coordinates and, using the lemma of S. L. Sobolev (^{1}), p. 474, obtain

$$B_2 = \iint_{\substack{r_1 \le r_2 \\ r_3 \le r_2}} \frac{f^*(r_1, \bar{y}_{n-s}) \varphi_2^*(r_2, \bar{x}_{n-s})}{r^\lambda} \, d\bar{x}_s \, d\bar{y}_s \le$$

$$\le c_1 \int_0^\infty \varphi_2^*(r_2, \bar{x}_{n-s}) r_2^{(s-1)/p'} F(r_2, \bar{y}_{n-s}) \left(\int_0^{r_2} \frac{r_3^{s-1} \, dr_3}{(r_3^2 + r_4^2)^{1/2\lambda}} \, dr_2 \right), \tag{6}$$

where

$$F(r_2, \bar{y}_{n-s}) = r_2^{-s/p' - 1/p} \int_0^{r_2} f^*(r_1, \bar{y}_{n-s}) r_1^{s-1} \, dr_1.$$

Taking into account inequality (6) and carrying out the necessary transformations, we shall have

$$\int_{R_y^{n-s}} B_i\, d\bar{y}_{n-s} \le c_1 \int_0^{\infty} \varphi_2^*(r_2, \bar{x}_{n-s}) r_2^{(s-1)/p'} \times$$

$$\times \left[\int_{R_y^{n-s}} F(r_2, \bar{y}_{n-s}) \left(\int_0^{r_2} \frac{r_3^{s-1}\, dr_3}{(r_3^2 + r_4^2)^{1/2\lambda}} \right) d\bar{y}_{n-s} \right] dr_2. \tag{7}$$

Since, by the hypothesis of the theorem, $p_2 \geqslant p_1 > p > 1$ and

$$\lambda = \frac{n-s}{p_2} + \frac{s}{p_1} + \frac{n}{p'} = (n-s)\left(\frac{1}{p_2} + \frac{1}{p'}\right) + s\left(\frac{1}{p_1} + \frac{1}{p'}\right),$$

after simple estimates we obtain

$$\int_0^{\infty} \varphi_2^*(r_2, \bar{x}_{n-s})\, r_2^{(s-1)/p'} \left[\int_{R_y^{n-s}} F(r_2, \bar{y}_{n-s}) \left(\int_0^{r_2} \frac{r_3^{s-1}\, dr_3}{(r_3^2 + r_4^2)^{1/2\lambda}} \right) d\bar{y}_{n-s} \right] dr_2$$

$$\leqslant c_2 \int_0^{\infty} \varphi_2^*(r_2, \bar{x}_{n-s})\, r^{\,s/p_1' - 1/p'} \left[\int_{R_y^{n-s}} F(r_2, \bar{y}_{n-s})\, r_4^{-(n-s)(1/p_2 + 1/p')}\, d\bar{y}_{n-s} \right] dr_2. \tag{8}$$

Let us now estimate the expression

$$I_2 = \int_{R_x^{n-s}} \varphi_1^{1/p_1}(\bar{x}_{n-s}) \left[\int_{R_y^{n-s}} B_2\, d\bar{y}_{n-s} \right] d\bar{x}_{n-s}. \tag{9}$$

Substituting in (9), in place of the expression in square brackets, its estimate from (7) and (8), and changing the order of integration in the resulting expression, we shall have

$$I_2 \leqslant c_3 \int_0^{\infty} r^{\,s/p_1' - 1/p'} \left[\int_{R_x^{n-s}} \varphi_1^{1/p_1}(\bar{x}_{n-s}) \varphi_2^*(r_2, \bar{x}_{n-s}) \right.$$

$$\left. \times \left(\int_{R_y^{n-s}} F(r_2, \bar{y}_{n-s})\, r_4^{-(n-s)(1/p_2 + 1/p')}\, d\bar{y}_{n-s} \right) d\bar{x}_{n-s} \right] dr_2.$$

To the integral standing in square brackets we apply Hölder’ s inequality. Taking into account the properties of integrals of potential type for the limiting exponent, we obtain

$$I_2 \leqslant c_3 \int_0^\infty r_2^{s/p_1'-1/p'} \left[\int_{R_x^{n-s}} \left| \varphi_1^{1/p_1}(\bar{x}_{n-s}) \varphi_2^*(r_2, \bar{x}_{n-s}) \right|^{p_2'} d\bar{x}_{n-s} \right]^{1/p_2'}$$

$$\times \left(\int_{R_y^{n-s}} |F(r_2, \bar{y}_{n-s})|^p \, d\bar{y}_{n-s} \right)^{1/p} dr_2. \qquad (10)$$

By hypothesis $p_2 > p_1 > p$ (for $p_2 = p_1$ the theorem was proved by S. L. Sobolev (^{1})); therefore we consider the positive numbers

$$\lambda = \frac{p_1(p_2-1)}{p_2-p_1}; \qquad \lambda' = \frac{p_1(p_2-1)}{p_2(p_1-1)} \qquad \left(\frac{1}{\lambda} + \frac{1}{\lambda'} = 1 \right)$$

and apply Hölder' s inequality to the integral standing in square brackets on the right-hand side of inequality (10), which, for convenience, we denote by $\bar{I}_2^{p_2'}$. Then

$$\bar{I}_2 \leqslant \|\varphi_1\|^{1/p_1}_{L_{(p_2/p_1)'}(R^{n-s})} \left\{ \int_{R_x^{n-s}} [\varphi_2^*(r_2, \bar{x}_{n-s})]^{p_1'} \, d\bar{x}_{n-s} \right\}^{1/p_1}. \qquad (11)$$

From (10), taking (11) into account, after elementary transformations we obtain

$$I_2 \leqslant c_4 \|\varphi_1\|^{1/p_1}_{L_{(p_2/p_1)'}(R^{n-s})} \int_0^\infty [\chi(r_2)]^{p/p_1} \Phi(r_2) \left[I_2^{1/p} \chi(r_2) \right]^{1-1/p_1} dr_2, \qquad (12)$$

where

$$\chi(r_2) = \left(\int_{R_y^{n-s}} |F(r_2, \bar{y}_{n-s})|^p \, d\bar{y}_{n-s} \right)^{1/p},$$

$$\Phi(r_2) = r_2^{(s-1)/p_1'} \left\{ \int_{R_x^{n-s}} [\varphi_2^*(r_2, \bar{x}_{n-s})]^{p_1'} \, d\bar{x}_{n-s} \right\}^{1/p_1'}.$$

Let us estimate the expression

$$r_2^{1/p} \chi(r_2) = r_2^{-s/p'} \left[\int_{R_y^{n-s}} \left(\int_0^{r_2} f^*(r_1, \bar{y}_{n-s}) \, r_1^{s-1} \, dr_1 \right)^p d\bar{y}_{n-s} \right]^{1/p}$$

$$= c_5 r_2^{-s/p'} \left[\int_{R_y^{n-s}} \left(\int_{C_{r_2}} f^*(r_1, \bar{y}_{n-s})\, d\bar{y}_s \right)^p d\bar{y}_{n-s} \right]^{1/p},$$

where C_{r_2} is the ball of radius r_2 in the s-dimensional space $(y_1, y_2, \dots, y_s)$ with center at the origin. Applying Hölder' s inequality to the inner integral and taking into account that $\|f^*\|_{L_p} = \|f\|_{L_p}$ (^{5}), we obtain the estimate

$$r_2^{1/p} \chi(r_2) \le c_6 \|f\|_{L_p(R^n)}; \tag{13}$$

therefore

$$I_2 \le c_7 \|\varphi_1\|^{1/p_1}_{L_{(p_2/p_1)'}} \|f\|^{1-p/p_1}_{L_p} \left\{ \int_0^\infty |\chi(r_2)|^p\, dr_2 \right\}^{1/p_1} \left\{ \int_0^\infty |\Phi(r_2)|^{p_1'}\, dr_2 \right\}^{1/p_1'}. \tag{14}$$

Let us estimate the integral

$$\int_0^\infty |\chi(r_2)|^p\, dr_2 = \int_{R_y^{n-s}} \left[\int_0^\infty r_2^{-(s/p'+1/p)} \left(\int_0^{r_2} f^*(r_1, \bar{y}_{n-s})\, r_1^{(s-1)/p} r_1^{(s-1)/p'}\, dr_1 \right)^p dr_2 \right] d\bar{y}_{n-s} \tag{15}$$

Using Hardy' s inequality (^{6}), for estimating the integral standing in the square brackets on the right-hand side of equality (15), we obtain

$$\int_0^\infty |\chi(r_2)|^p\, dr_2 \le c_8 \|f\|^p_{L_p(R^n)}. \tag{16}$$

It is also not difficult to show that

$$\int_0^\infty |\Phi(r_2)|^{p_1'}\, dr_2 \le c_9 \|\varphi_2\|^{p_1'}_{L_{p_1'}(R^n)}. \tag{17}$$

From (14), taking (16) and (17) into account, we shall have

$$I_2 \le c_{10} \|f\|_{L_p(R^n)} \|\varphi_2\|_{L_{p_1'}(R^n)} \|\varphi_1\|^{1/p_1}_{L_{(p_2/p_1)'}(R^{n-s})}.$$

Theorem 2 is completely proved.

I take this opportunity to express my gratitude to S. V. Uspenskii for the valuable comments he made in carrying out this work.

Institute of Mathematics with Computing Center
of the Siberian Branch of the Academy of Sciences of the USSR

Received
6 VII 1962

REFERENCES

[1] S. L. Sobolev, *Matem. sborn.*, **4** (46), No. 3 (1938).
[2] V. I. Kondrashev, *DAN*, **48**, 563 (1945).
[3] V. P. Il’ in, *DAN*, **96**, 905 (1954).
[4] L. V. Kantorovich, *UMN*, **11**, 2 (68), 3 (1956).
[5] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, 1948.
[6] G. Pólya, G. Szegő, *Isoperimetric Inequalities in Mathematical Physics*, 1962.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.