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Abstract

Full Text

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On the Solvability Condition for the Homogeneous Riemann Problem on Closed Riemann Surfaces

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1. Let on a closed Riemann surface R of genus ρ there be given a contour Γ, consisting of a finite number of mutually nonintersecting smooth closed curves

$$\Gamma = \bigcup_{j=1}^{m} \Gamma_j.$$

Consider the Riemann boundary-value problem

$$\Phi^+(P) = G(P)\Phi^-(P) \tag{1}$$

under the assumption that $G(P) \in H(\Gamma)$, $G(P) \neq 0$.

The question of the solvability of problem (1) depending on the value of the index

$$\varkappa = \frac{1}{2\pi i}\int_\Gamma d\ln G = \sum_{j=1}^{m} \frac{1}{2\pi i}\int_{\Gamma_j} d\ln G = \sum_{j=1}^{m} \varkappa_j \tag{2}$$

was studied in works ([4–6]), where it was proved that for $\varkappa \geq \rho$ the problem is always solvable (a regular solution is meant, not identically equal to zero), while for $\varkappa < 0$ there are no regular solutions except $\Phi \equiv 0$.

In the present paper we give a necessary and sufficient condition for the solvability of problem (1) when $0 \leq \varkappa < \rho$, in terms of Riemann ϑ-functions, supplementing the results known for this case from work ([5]).

2. Choose on each contour Γ_j one point P_j $(j = 1, 2, \dots, m)$ as the initial point for traversing the contour, and an arbitrary branch $\ln G$, single-valued on the contour Γ cut at the points P_j $(j = 1, 2, \dots, m)$. Consider the function

$$\Psi(z) = \exp\left\{\frac{1}{2\pi i}\int_\Gamma \ln G(t)\, A^*(t,z)\, dt\right\}, \tag{3}$$

where $A^*(\zeta, z)$ is the multivalued Cauchy kernel with periods

$$\int_{A_\nu} d_z A^*(\zeta, z) = 0, \qquad \int_{B_\nu} d_z A^*(\zeta, z) = 2\pi i\,\frac{dw_\nu}{d\zeta} \quad (\nu = 1, 2, \dots, \rho), \tag{4}$$

and $dw_1, \dots, dw_\rho$ is a complex-normalized basis of Abelian differentials of the first kind ([7]). From (4) and the known properties of an integral of Cauchy type ([1]) it follows that $\Psi(z)$ is a multiplicatively multivalued function with multipliers

$$\mu(A_\nu) = 1, \qquad \mu(B_\nu) = \exp \int_\Gamma \ln G \, dw_\nu \quad (\nu = 1, \dots, \rho), \tag{5}$$

has order $\varkappa_j$ at the point P_j $(j = 1, 2, \dots, m)$, and everywhere on Γ satisfies condition (1).

Theorem 1. *For problem (1) to be solvable it is necessary and sufficient that on the surface R there exist $\varkappa$ points $R_1, \dots, R_\varkappa$ (not necessarily distinct) for which the system of congruences*

$$\sum_{j=1}^{\varkappa} \int_{P_0}^{R_j} dw_\nu \equiv -\frac{1}{2\pi i} \int_\Gamma \ln G \, dw_\nu + \sum_{j=1}^{m} \varkappa_j \int_{P_0}^{P_j} dw_\nu \quad \text{(modulo periods)} \tag{6}$$

$$(\nu = 1, \dots, \rho); \qquad P_0 \text{ is an arbitrarily fixed point.}$$

Necessity. Denote the solution of the problem by Φ and its zeros by $R_1, \dots, R_\chi$. The function $F = \Psi/\Phi$, where Ψ is given by formula (3), is multiplicatively multivalued with multipliers (5), meromorphic on the whole surface, having order χ_j at the point P_j $(j = 1, 2, \dots, m)$ and poles at the points $R_1, \dots, R_\chi$. Consequently, $d \ln G$ is a differential of the third kind with periods

$$\pi(B_\nu) = \int_\Gamma \ln G \, d\omega_\nu + 2\pi i m_\nu, \qquad \pi(A_0) = 2\pi i l_\nu,$$

where m_ν, l_ν are integers $(\nu = 1, 2, \dots, \rho)$, and with poles at the points $P_1, \dots, P_m, R_1, \dots, R_\chi$. From the known relation between periods (([2]), Theorem 10.6) we obtain (6).

Sufficiency. To prove sufficiency, consider the differential of the third kind

$$d\Omega = -\sum_{j=1}^{m} \chi_j d\omega_{P_j P_0} + \sum_{j=1}^{\chi} d\omega_{R_j P_0},$$

where $d\omega_{PQ}$ is a complex-normalized differential of the third kind with residues $+1$ and -1 at the points P and Q, respectively, and the points taken as R_j $(j = 1, \dots, \chi)$ are points satisfying system (6). On the basis of (6) and the above-mentioned relation between periods (([2]), Theorem 10.6), we verify that $F_1 = \exp \int d\Omega$ is a multiplicatively multivalued function with multipliers $1/\mu_\nu$ $(\nu = 1, 2, \dots, \rho)$ and order $-\chi_j$ at the point P_j $(j = 1, 2, \dots, m)$. Then, as is easy to check, $\Phi = \Psi F_1$ will be a solution of the problem with zeros at the points $R_1, \dots, R_\chi$ (see, for example, ([1]), §44).

Let us note that the choice of the points P_j $(j = 1, 2, \dots, m)$ and of the branch of $\ln G$ does not affect the solvability of system (6), as is easily verified by a simple calculation.

3. Denoting

$$\lambda_\nu = -\frac{1}{2\pi i}\int_\Gamma \ln G\, d\omega_\nu + \sum_{j=1}^{m} \chi_j \int_{P_0}^{P_j} d\omega_\nu, \qquad u_\nu(P) = \int_{P_0}^{P} d\omega_\nu \quad (\nu = 1, 2, \dots, \rho), \tag{7}$$

we rewrite system (6) in the form

$$\sum_{j=1}^{\chi} u_\nu(R_j) \equiv \lambda_\nu \quad \text{(modulo periods)} \tag{8}$$

$$(\nu = 1, 2, \dots, \rho).$$

The problem of finding points $R_1, \dots, R_\chi$ satisfying system (8) for a given right-hand side is known as the Jacobi inversion problem. It was first applied to the solution of the Riemann problem in the work ([5]). For $\chi \geq \rho$ the inversion problem is always solvable, which agrees with the results of ([4–6]).

To study the question of solvability of system (8) for $0 \leq \chi < \rho$, we turn to the Riemann theta function of ρ variables $u_1(P), u_2(P), \dots, u_\rho(P)$, denoting it briefly by $\vartheta(u_1, \dots, u_\rho) = \vartheta(u_\nu)$. We shall need the following properties of it ([3]):

I. ϑ is an even function.

II.

$$\vartheta\left(\sum_{j=1}^{\rho-1} u_\nu(Q_j) + k_\nu\right) = 0$$

for any divisor $Q_1, Q_2, \dots, Q_{\rho-1}$, where k_ν $(\nu = 1, 2, \dots, \rho)$ is a system of numbers depending only on the surface.

III. Whatever the numbers e_ν $(\nu = 1, 2, \dots, \rho)$, there exists a divisor $Q_1^0, Q_2^0, \dots, Q_\rho^0$ such that

$$\vartheta\left(\sum_{j=1}^{\rho} u_\nu(Q_j^0) + e_\nu + k_\nu\right) \neq 0.$$

IV. If $\vartheta(u_\nu(P) - e_\nu) \not\equiv 0$ as a function of the point P, then it has ρ zeros on the surface R, and if these are denoted by $Q_1', Q_2', \dots, Q_\rho'$,

the following system of congruences will hold:

$$e_\nu \equiv \sum_{j=1}^{\rho} u_\nu\left(Q_j'\right) + k_\nu \quad \text{(modulo periods)}$$

$$(\nu = 1, \dots, \rho).$$

Theorem 2. *For the solvability of system (8) it is necessary and sufficient that*

$$\vartheta\left(\sum_{j=1}^{\rho-\chi-1} u_\nu(Q_j)+\lambda_\nu+k_\nu\right)=0 \tag{9}$$

for any divisor $Q_1,\dots,Q_{\rho-\chi-1}$.

Necessity follows from property II.

Sufficiency. To prove sufficiency, denote by l the greatest integer for which

$$\vartheta\left(\sum_{j=1}^{\rho-\chi+l} u_\nu(Q_j)+\lambda_\nu+k_\nu\right)=0 \tag{10}$$

for any divisor $Q_1,\dots,Q_{\rho-\chi+l}$. In view of (9) and property III, such a number exists and $-1\le l\le\chi-1$. Let $Q_1^0,\dots,Q_{\rho-\chi+l+1}^0$ be a divisor for which

$$\vartheta\left(\sum_{j=1}^{\rho-\chi+l+1} u_\nu(Q_j^0)+\lambda_\nu+k_\nu\right)\neq 0$$

and, consequently, the function

$$\vartheta\left(u_\nu(P)-\sum_{j=1}^{\rho-\chi+l+1} u_\nu(Q_j^0)-\lambda_\nu-k_\nu\right) \tag{11}$$

is not identically equal to zero. But, in view of (10),

$$\vartheta\left(u_\nu(Q_i^0)-\sum_{j=1}^{\rho-\chi+l+1} u_\nu(Q_j^0)-\lambda_\nu-k_\nu\right)=0 \quad (j=1,\dots,\rho-\chi+l+1) \tag{12}$$

and, using property IV, we are convinced of the validity of system (8), if the zeros of the function (11), distinct from $Q_1,\dots,Q_{\rho-\chi+l+1}$, are taken as the points $R_1,\dots,R_{\chi-l+1}$, and $R_{\chi-l}=R_{\chi-l+1}=\dots=R_\chi=P_0$.

As an application of Theorem 2, we point out that equality (9) gives a necessary and sufficient condition for the solvability of Hilbert's problem $\mathrm{Re}[(a-ib)F]=0$ on a finite Riemann surface, in particular on a plane multiply connected domain, if it is reduced by the known method (4) to problem (1) on the doubled Riemann surface.

If, for $\chi<0$, as a solution of problem (1) one allows functions having $|\chi|$ poles on the surface, then the condition for solvability of the problem will likewise be given by Theorem 2.

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Note: Figure translations are in progress. See original paper for figures.