MATHEMATICS

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1963

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Abstract

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ON THE THEORY OF GENERALIZED GROUPS

(Presented by Academician A. I. Mal' tsev, 5 VI 1963)

An element g of a semigroup G is called **generalized invertible** if, for some $\bar{g} \in G$, $g\bar{g}g = g$, $\bar{g}g\bar{g} = \bar{g}$. In this case the element $\bar{g}$ is called a **generalized inverse** for g. It is known ([1]) that an element g is generalized invertible if, for some $g' \in G$, $gg'g = g$.

We shall call a semigroup **weakly regular** if the product of any two idempotents of this semigroup is generalized invertible, and **regular** if every element of this semigroup is generalized invertible. We shall call a semigroup G **singular** if it contains more than one element and, for any two elements $g_1, g_2 \in G$, $g_1g_2 = g_1$, or else, for any two elements $g_1, g_2 \in G$, $g_1g_2 = g_2$.

V. V. Wagner introduced the notion of a **generalized group** ([2]), defining it as a regular semigroup satisfying the property

A1. *Any two idempotents of the semigroup commute under multiplication.*

Generalized groups were introduced by G. Preston ([3]), who called them "inverse semigroups." The same name is used in monographs on the theory of semigroups ([4,5]).

Let us consider the following conditions which semigroups may satisfy:

A2. *For any element g and any generalized inverse $\bar{g}$ of it, the elements $g\bar{g}$ and $\bar{g}g$ commute under multiplication.*

A3. *For each element of the semigroup there exists at most one generalized inverse.*

A4. *For each idempotent of the semigroup there exists at most one generalized inverse.*

A4′. *For each idempotent of the semigroup there exists at most one generalized inverse that is an idempotent.*

A5. *For any two idempotents i_1, i_2,*

$$i_1i_2 = i_1 \leftrightarrow i_2i_1 = i_1.$$

A6. *Every idempotent commutes under multiplication with any of its generalized inverses.*

A6′. *Every idempotent commutes under multiplication with any of its idempotent generalized inverses.*

A7. *If i is an idempotent and $\bar{i}$ is its generalized inverse, then the elements $\bar{i}i$ and $i\bar{i}$ commute under multiplication.*

A7′. *If i is an idempotent and $\bar{i}$ is its idempotent generalized inverse, then the elements $\bar{i}i$ and $i\bar{i}$ commute under multiplication.*

A8. *If i_1, i_2 are idempotents, and $i_1 i_2 = i_1$, $i_2 i_1 = i_2$, then $i_1 = i_2$; if $i_1 i_2 = i_2$, $i_2 i_1 = i_1$, then $i_1 = i_2$.*

Theorem 1. *From the satisfaction of condition A1 follows the satisfaction of condition A2; from the satisfaction of condition A2 follows the satisfaction of condition A3. Conditions A3, A4, A4′, A5, A6, A6′, A7, A7′, A8 are equivalent. Satisfaction of condition A3, generally speaking, does not imply satisfaction of condition A2, and satisfaction of A2 does not imply satisfaction of A1.*

Proof. Since $g\bar{g}$ and $\bar{g}g$ are idempotents if $\bar{g}$ is a generalized inverse for g, A2 follows from A1. A3 is obviously stronger than A4. Suppose A4′ is satisfied and, for idempotents i_1, i_2, $i_1 i_2 = i_1$. Then i_1 and $i_2 i_1$ will be generalized inverses for i_1, and i_1 and $i_2 i_1$ will be

idempotents. Therefore $i_2 i_1 = i_1$. Similarly, from $i_2 i_1 = i_1$ it follows that $i_1 i_2 = i_1$, i.e. A5 holds. It is obvious that A4′ follows from A4. Suppose A5 holds and $\bar{i}$ is a generalized inverse for the idempotent i. Then $i = i(\bar{i}i) = (\bar{i}i)i = \bar{i}i$ and $i = (i\bar{i})i = i(i\bar{i}) = i\bar{i}$, whence $\bar{i} = i\bar{i}$, i.e. A6 holds. From A6, A6′ follows in an obvious way. Suppose A6′ holds and the idempotent $\bar{i}$ is a generalized inverse for the idempotent i. Then $\bar{i}i\bar{i} = \overline{iii}$, i.e. A7′ holds. Suppose A7′ holds and, for the idempotents i_1, i_2, $i_1 i_2 = i_1$, $i_2 i_1 = i_2$. Multiply the first equality on the right by i_1, and the second on the right by i_2. Then it is seen that i_2 will be a generalized inverse for i_1. By condition A7′ the elements $i_1 = i_1 i_2$ and $i_2 = i_2 i_1$ commute, whence $i_1 = i_2$. Thus A8 holds. We note that A7′ follows in an obvious way from A7. If, in the proof that A7′ follows from A6′, $\bar{i}$ is regarded as an arbitrary generalized inverse for i, then one obtains a proof that A7 follows from A6. Finally, suppose A8 holds and the elements $\bar{g}, \bar{\bar{g}}$ are generalized inverses for g. Then $(g\bar{g})(g\bar{\bar{g}}) = g\bar{\bar{g}}$, $(g\bar{\bar{g}})(g\bar{g}) = g\bar{g}$, whence $g\bar{g} = g\bar{\bar{g}}$. Similarly, $\bar{g}g = \bar{\bar{g}}g$. Therefore $\bar{g} = \bar{g}gg = \bar{\bar{g}}g\bar{g} = \bar{\bar{g}}gg = \bar{\bar{g}}$, i.e. A3 holds. Consequently, conditions A3—A8 are equivalent. Since A7 follows in an obvious way from A2, A3 follows from A2. To prove that A2 does not follow from A3, and that A1 does not follow from A2, the corresponding examples are constructed, which we omit.

Theorem 2. *In a weakly regular semigroup the properties A1—A8 are equivalent.*

Proof. Let G be a weakly regular semigroup satisfying condition A3, and let i_1, i_2 be idempotents. Then there exists an element g, generalized inverse for $i_1 i_2$. We next show that gi_1 and $i_2 g$ will also be generalized inverses for $i_1 i_2$, whence $g = gi_1 = i_2 g$. From this we derive that g is an idempotent. By A3, $i_1 i_2 = g$, whence $i_1 i_2 = i_1 i_2 i_1 = i_2 i_1 i_2$. Similarly $i_2 i_1 = i_2 i_1 i_2 = i_1 i_2 i_1$, i.e. $i_1 i_2 = i_2 i_1$, and condition A1 holds. The assertion of Theorem 2 now follows from Theorem 1.

Corollary. *In a regular semigroup the properties A1—A8 are equivalent.*

It follows from this that, in order for a semigroup to be a generalized group, it is necessary and sufficient that this semigroup be regular and satisfy any one of the properties A1—A8.

A. E. Liber [(6)] proved that the properties A1 and A3 are equivalent in regular semigroups; W. Mann and R. Penrose [(7)] proved that in regular semigroups the properties A1, A3, and A4 are equivalent.

We note that condition A8 can be replaced by the following equivalent nonelementary condition: *the semigroup contains no singular subsemigroups.*

An **involutory semigroup** is a semigroup on which an **involution** is given, i.e. a mapping which associates with each element g of the semigroup an element g^{-1}, where $(g^{-1})^{-1} = g$ and $(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$. An involutory semigroup will be called a generalized group if it is a generalized group and g^{-1} is a generalized inverse for g. From this point of view, a generalized group is an algebra with two operations—multiplication and involution.

Theorem 3. *An involutory semigroup will be a generalized group if and only if the following two identities hold in it:*

B1. $gg^{-1}g = g$.

B2. $gg^{-1}g^{-1}g = g^{-1}ggg^{-1}$.

The class of generalized groups cannot be singled out from the class of involutory semigroups by a single identity.

From this theorem, whose proof we omit, it follows that the class of generalized groups, considered as involutory semigroups, is primitive (for the definition of primitivity, see [(8)]).

For generalized groups considered as algebras with two operations, one can construct new operations derived from multiplication and in-

involution. These operations are expressed by polynomials in the two initial operations of generalized groups (for the definition of a polynomial see, for example, ([9])). A derived operation will be called **mutual** if the basic operations —multiplication and involution—in generalized groups can in turn be expressed as derived operations from this derived operation. It is known that for the class of groups there exist mutual binary operations (such are the operations of right and left division ([10])); for the class of rings there are no mutual binary operations, but there do exist mutual ternary operations ([11]); for the class of Boolean algebras there exist mutual binary and ternary operations (the Sheffer stroke, the median operation ([12])).

Theorem 4. *For the class of generalized groups there are no mutual operations.*

Let us note that condition B2 of Theorem 3 can be replaced by the following elementary condition, which is not an identity: *the involution leaves every idempotent fixed.*

An **ordered semigroup** is a semigroup on which a stable order relation is given ([4]). We shall call an ordered semigroup a generalized group if this semigroup is a generalized group and its order relation coincides with the canonical order relation of the generalized group ([2]).

An order relation $<$ is called **stable** ([13]) if it satisfies the condition $z < xv,\ uv,\ uy \to z < xy$.

Theorem 5. *An ordered semigroup is a generalized group if and only if it is regular and its order relation is stable.*

An **involuted ordered semigroup** is an ordered semigroup on which an involution is given and whose order relation is invariant under involution, i.e. $g_1 < g_2 \to g_1^{-1} < g_2^{-1}$.

Theorem 6. *An involuted ordered semigroup is a generalized group if and only if it satisfies condition B1 and the condition*

C1. *If i is an idempotent, then $gi < g$ for every g.*

By a **groupoid** we shall mean a set on which a partial binary operation is defined and with respect to which our set is a category all of whose elements are invertible (for a more detailed definition see, e.g., P. Croisot ([14]); he was apparently the first to consider such systems, or ([15], [16])). Brandt groupoids are a special case of groupoids. Let us note that we define the structure of a groupoid on a set, and not on a class, solely because of possible set-theoretic complications.

Define on a generalized group G a partial binary operation, defined for elements g_1 and g_2 if and only if $g^{-1}g_1 = g_2g^{-1}$ (g^{-1} denotes the generalized inverse of g) and coinciding with the multiplication operation of the generalized group in those cases where this partial operation is defined. The operation obtained will be called **adjacent multiplication**. The set G with the operation of adjacent multiplication introduced on it turns out to be a groupoid. This groupoid is called the groupoid **associated** with the generalized group G. A groupoid is called **completable** if it is associated with some generalized group. A groupoid is completable if and only if the operation of this groupoid can be extended to an everywhere-defined operation in such a way that the groupoid becomes a semigroup, the units of the groupoid commute under multiplication (i.e. the groupoid becomes a generalized group), and no new elements are adjoined to the groupoid.

It can be shown that every groupoid G can be uniquely decomposed into the union of pairwise disjoint subsets that are Brandt groupoids with respect to the operation induced on themtions, and such that the product of a pair of elements taken from different subsets is never defined. We shall call the groupoid G a union of these Brandt groupoids.

Theorem 7. *A groupoid is completable if and only if the set of units of this groupoid is infinite, or if this groupoid is a union of Brandt groupoids, one of which is a group. In particular, a Brandt groupoid is completable if and only if it is a group or if it contains an infinite set of units.*

It follows from this that a groupoid is not completable if and only if it is a union of a finite number of Brandt groupoids, none of which is a group. In general, the completion of a groupoid is not unique.

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Received
25 V 1963

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Note: Figure translations are in progress. See original paper for figures.