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Abstract

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MATHEMATICS

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PROPAGATION OF SMOOTHNESS OF SOLUTIONS OF DIFFERENTIAL EQUATIONS OF PRINCIPAL TYPE

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Various authors have recently established a number of results concerning the continuation of smoothness of solutions of an arbitrary partial differential equation with constant coefficients

$$P\left(\frac{1}{i}\frac{\partial}{\partial x_1}, \dots, \frac{1}{i}\frac{\partial}{\partial x_n}\right) u(x_1, \dots, x_n) = 0. \tag{1}$$

Thus, M. S. Agranovich in ([1]) proved that every solution of equation (1), infinitely differentiable in a neighborhood of the boundary of a bounded domain V, is automatically infinitely differentiable throughout the whole domain V. In works ([2–5]) this result was extended as follows: in the case of a domain of the form $V \setminus W$, where W is a convex closed set, it is sufficient to require infinite differentiability of the function $u(x)$ only in a neighborhood of the set $\dot{V} \setminus W$ ($\dot{V}$ is the boundary of the domain V). Now the following question naturally arises: how can one describe all closed subsets belonging to V and having the property that any solution $u(x)$ of equation (1) in the domain V, which is infinitely differentiable in a neighborhood of such a subset, will be infinitely differentiable in some neighborhood of the point $a \in V$? Here we shall give a complete description of such subsets for the case when the domain V is star-shaped with respect to the point a and equation (1) satisfies the following two conditions.

Condition I. The characteristic polynomial $P(\sigma_1, \dots, \sigma_n)$, which is obtained from equation (1) by replacing $\frac{1}{i}\frac{\partial}{\partial x_j}$ by σ_j, has real coefficients.*

Condition II. For every nonzero real $\sigma = (\sigma_1, \dots, \sigma_n)$,
$\operatorname{grad} P_0(\sigma) \neq 0$, where $P_0(\sigma)$ is the principal part of the polynomial $P(\sigma)$.

In order to formulate the main result, we shall need the concept of a bicharacteristic. The cone $P_0(\sigma) = 0$ is called **characteristic**. From Conditions I and II it follows that this cone has dimension $n-1$. A **bicharacteristic** is any straight line in the space $R^n(x)$ passing through the origin parallel to some normal to the characteristic cone.

Theorem 1. *Let equation* (1) *satisfy Conditions I and II, and let V be a bounded domain, star-shaped with respect to the origin. In order that, in the domain V, every solution of equation* (1) *which is infinitely differentiable in a neighborhood of the closed set $W \subset \dot{V}$ be infinitely differentiable in some neighborhood of the origin, it is necessary and sufficient that every bicharacteristic have at least one point in common with the set W.*

The necessity of this condition follows directly from the existence of a solution of equation (1) which has singularities on a given bicharacteristic. Such solutions were constructed by L. Hörmander in ([4]).

* Concerning this condition, see Remark 3 at the end of the paper.

To prove sufficiency, consider a function $\alpha(x)$ that is equal to zero outside the domain V and equal to one in $V \setminus \dot{V}_\varepsilon$, where $\dot{V}_\varepsilon$ is the ε-neighborhood of the boundary of the domain V. Then for any solution of equation (1), for sufficiently small x we shall have

$$u(x) = u(x)\alpha(x) * \delta = P(D)[u(x)\alpha(x)] * \mathcal{E}(x),$$

where $\mathcal{E}(x)$ is a fundamental solution of equation (1). Thus everything depends on a successful choice of the fundamental solution. In order to obtain the result of M. S. Agranovich, one may use any fundamental solution. To obtain the more precise result stated above, it is necessary to have a fundamental solution sufficiently smooth in a half-space.* If it is possible to choose $\mathcal{E}(x)$ so that in the domain $\dot{V}_\varepsilon \setminus W_\varepsilon$ $\mathcal{E}(-x)$ is an infinitely differentiable function, then the assertion of the theorem will be proved.

Theorem 2. *Let B be a set by which the cone of bicharacteristics intersects the unit sphere Ω, and let H be a closed subset of the set B such that any bicharacteristic has at most one common point with H. If conditions I and II are satisfied for equation (1), then there exists a fundamental solution that is infinitely differentiable outside the cone formed by the rays joining the origin to the points of the set $B \setminus H$.*

Construction of the fundamental solution. We first consider a homogeneous differential equation of order m. Fundamental solutions for such equations were constructed by V. A. Borovikov (see ([6,7]), p. 165) in the form

$$\mathcal{E}(x) = \frac{1}{(2\pi)^n} \int_\Omega \frac{f_{mn}(x,\omega)\, d\omega}{P(\omega_1,\dots,\omega_n)}, \tag{2}$$

where ω is a vector of the unit sphere, $(x, \omega) = x_1\omega_1 + \dots + x_n\omega_n$, and f_{mn} is a specially chosen function. For simplicity we shall henceforth assume that $m > n$, since this can always be achieved by multiplying equation (1) by a suitable power of the Laplace operator. In this case, as $f_{mn}(t)$ we shall take

$$f_{mn}(t) = -\frac{i^{m-n}}{(m-n)!}\, t^{m-n} \ln(t + i0). \tag{3}$$

Now everything depends on the choice of a regularization of the divergent integral (2). Regularization in the sense of the principal value is not suitable for us, since in this case one obtains a fundamental solution with singularities on the entire cone of bicharacteristics.

Let the unit sphere Ω contain two closed nonintersecting sets Λ_+ and Λ_-, and let the function $\chi(\omega)$ be equal to $+1$ in a neighborhood of the set Λ_+ and equal to -1 in a neighborhood of Λ_-. Take a sufficiently fine partition of unity on the sphere Ω, $\sum \varphi_k = 1$, and consider

$$\mathcal{E}_k(x) = \frac{1}{(2\pi)^n} \int_\Omega \frac{\varphi_k(\omega) f_{mn}(x, \omega)\, d\omega}{P(\omega_1, \dots, \omega_n)}. \tag{4}$$

If in a neighborhood of the support of the function $\varphi_k(\omega)$ the polynomial $P(\omega) \neq 0$, then expression (4) requires no further explanation. If, however, this is not the case, then in the neighborhood under consideration we can introduce coordinates $u_1, \dots, u_{n-1}$ such that the surface $P(\omega) = 0$ in these coordinates will have equation $u_1 = 0$, and the polynomial $P(\omega)$ will be written in the form $P(\omega) = u_1 f(u_1, \dots, u_{n-1})$,

* The corresponding theorem in ([5]) is formulated not quite precisely. We give the correct formulation of the assertion proved in ([5]): for an arbitrary vector η and any two positive numbers r and δ there exists a fundamental solution of equation (1) which in the half-space $(x, \eta) > \delta$ is an ordinary $[r]$-times continuously differentiable function.

where $f(u_1, \dots, u_{n-1}) \neq 0$. Put

$$\mathcal{E}_k(x) = \frac{1}{(2\pi)^n} \int \cdots \int \left\{ \int^* \frac{\varphi_k(u) f_{mn}[x, \omega(u)]}{u_1 f(u)} D\binom{\omega}{u}\, du_1 - i\pi \frac{\chi(u)\varphi_k(u) f_{mn}[x, \omega(u)]}{f(u)} D\binom{\omega}{u}\Bigg|_{u_1=0} \right\} du_2 \cdots du_{n-1} \tag{5}$$

where the inner integral is taken in the sense of the principal value. After this we find $\mathcal{E}(x)$ as $\sum \mathcal{E}_k(x)$.

It is comparatively easy, by integration by parts, to show that expression (5) is an infinitely differentiable function outside the cone of bicharacteristics. The proof of the smoothness of the function $\mathcal{E}_k(x)$ in a neighborhood of points of H is based on the following lemma.

Lemma. Let $\varphi(x,t)$ and $v(x,t)$ be infinitely differentiable functions. If $\varphi(x,t)$ is finite and $v_t(x,t) > 0$, then

$$\int^{*} \frac{\varphi(x,t) f_{mn}[v(x,t)]}{t}\, dt - i\pi\varphi(x,0) f_{mn}[v(x,0)] \tag{6}$$

is an infinitely differentiable function of x.

We note that in the case when φ and v are analytic functions in some neighborhood of the origin, in integral (6) the integration along the real axis may be replaced by integration over a complex contour, after which the assertion of the lemma becomes obvious.

Choosing the sets Λ_+ and Λ_- in an appropriate way, we obtain the fundamental solution $\mathcal{E}(x)$ that we need.

Construction of fundamental solutions of nonhomogeneous equations by the descent method. We shall first construct a function $F(x)$ satisfying the equation

$$P\left(\frac{1}{i}\frac{\partial}{\partial x}\right) F(x) = (-\Delta)^m \delta.$$

Solving then the equation $(-\Delta)^m \mathcal{E}(x) = F(x)$, one can find a fundamental solution of our equation, and its singularities will be located where the singularities of $F(x)$ are located.

Let us outline the plan for constructing $F(x)$. It is well known that $F(x)$ can be obtained as the Fourier transform of $|\sigma|^{2m}/P(\sigma)$. Represent the polynomial $P(\sigma)$ in the form $P_0(\sigma) + P_1(\sigma) + \dots + P_m(\sigma)$, where $P_j(\sigma)$ is a homogeneous polynomial of order $m-j$, and consider the homogeneous polynomial in $n+1$ variables

$$\Pi(\sigma_1, \dots, \sigma_n, \sigma_{n+1}) = P_0(\sigma) + \sigma_{n+1} P_1(\sigma) + \dots + \sigma_{n+1}^m P_m(\sigma).$$

Obviously,

$$P(\sigma_1, \dots, \sigma_n) = \Pi(\sigma_1, \dots, \sigma_{n+1})\big|_{\sigma_{n+1}=1}.$$

If it is known that the Fourier transform of

$$\frac{(\sigma_1^2 + \dots + \sigma_n^2)^m}{(2\pi)^{n+1}\Pi(\sigma_1, \dots, \sigma_{n+1})}$$

is $\Phi(x_1, \dots, x_{n+1})$, then we may try to find the function $F(x)$ in the form

$$F(x) = \int \Phi(x_1, \dots, x_{n+1}) e^{-ix_{n+1}}\, dx_{n+1}. \tag{7}$$

The function $\Phi(x_1, \dots, x_{n+1})$ is readily found from the formula

$$\Phi(x_1, \dots, x_{n+1}) = \frac{(-1)^m}{(2\pi)^{n+1}} \left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} \right)^m \int_{\Omega_{n+1}} \frac{f_{mn+1}(\bar{x}, \bar{\omega})}{\Pi(\omega_1, \dots, \omega_{n+1})} \, d\bar{\omega}, \quad (8)$$

where $\bar{x}$ and $\bar{\omega}$ denote $(n+1)$-dimensional vectors. From this formula it is clear that Φ is a homogeneous function of order $-m-n-1$; therefore no difficulties with the convergence of expression (7) at infinity arise.

Let now $\psi_\varepsilon(\bar{\omega})$ be a function equal to one in the ε-neighborhood of the set $\omega_{n+1} = 0$ and equal to zero outside the 2ε-neighborhood.

Consider

$$\Phi_\varepsilon(x_1, \dots, x_{n+1}) = \frac{(-1)^m}{(2\pi)^{n+1}} \left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} \right)^m \int_{\Omega_{n+1}} \frac{\psi_\varepsilon(\bar{\omega}) \, f_{mn+1}(\bar{x}, \bar{\omega})}{\Pi(\omega_1, \dots, \omega_{n+1})} \, d\bar{\omega}, \quad (9)$$

$$F_\varepsilon(x_1, \dots, x_n) = \int \Phi_\varepsilon(x_1, \dots, x_{n+1}) e^{-i x_{n+1}} \, dx_{n+1}. \quad (10)$$

From these relations it is clear that $F_\varepsilon(x)$ is the Fourier transform of

$$\left. \frac{\varphi(\bar{\omega})(\sigma_1^2 + \dots + \sigma_n^2)^m}{(2\pi)^n \Pi(\sigma_1, \dots, \sigma_{n+1})} \right|_{\sigma_{n+1}=1},$$

and, since this function coincides with $|\sigma|^{2m}/P(\sigma)$ for sufficiently large $|\sigma|$, $F_\varepsilon(x)$ differs from $F(x)$ by an analytic function. Therefore the singularities of $F(x)$ are located where the singularities of $F_\varepsilon(x)$ are located. But from relations (9) and (10) it is clear that, for small ε, the singularities of $F_\varepsilon(x)$ lie in a sufficiently small conical neighborhood of the cone of characteristics; hence we conclude that $F(x)$ is infinitely differentiable outside this cone. Moreover, by choosing in a suitable way the regularization of the integrals (8) and (9), one can arrange that the singularities of the function $F(x)$ fill not the whole cone of characteristics, but only some part of it. In this way one can obtain a fundamental solution satisfying all the requirements of Theorem 2. All the arguments given above can be justified quite rigorously with the aid of the theory of generalized functions.

Remark 1. From our formulas it is clear that $\mathcal{E}(x)$ is an $m-n-2$ times continuously differentiable function when $m > n+1$, and is a combination of derivatives of order $n-m+2$ of continuous functions when $m \le n+1$. Therefore, if the conditions of Theorem 1 are fulfilled, one may assert that every solution $u(x)$ which has, in a neighborhood of the set W, locally summable derivatives up to order k inclusive ($k \geq n+1$) will be, in some neighborhood of the origin, $k-n-1$ times continuously differentiable (for homogeneous equations, $k-n$ times).

Remark 2. Theorem 1, for equations satisfying conditions I and II, answers the following question posed by B. Malgrange in ([3]): will every solution of equation (1), infinitely differentiable in the half-space $(x, \eta) > 0$, automatically be infinitely differentiable in the whole space? The answer is affirmative if the plane $(x, \eta) = 0$ contains no characteristic.

Remark 3. All assertions of this paper are also valid for equations with complex coefficients for which the surface

$$\Pi(\sigma_1, \dots, \sigma_{n+1}) = 0$$

in some neighborhood of the form

$$\sigma_{n+1}^2 < \varepsilon(\sigma_1^2 + \dots + \sigma_n^2)$$

has dimension n at each of its points.

Remark 4. It can be shown that if, to an operator of order m satisfying conditions I and II, one adds lower-order terms of order $\leq m-2$ with arbitrary (even complex) polynomial coefficients, then the assertion of Theorem 1 remains valid.

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Note: Figure translations are in progress. See original paper for figures.