ON WEAK COMPLETE CONTINUITY OF A LINEAR MAPPING AND OF ITS ADJOINT

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Abstract

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ON WEAK COMPLETE CONTINUITY OF A LINEAR MAPPING AND OF ITS ADJOINT

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V. Gantmacher (^1) proved that a linear mapping of a Banach space X into a Banach space Y is weakly completely continuous if and only if the adjoint mapping is weakly completely continuous. The present note is devoted to extending each of the two constituent parts of this result to broader classes of spaces. For the definitions of the concepts used here, see (^2). X and Y will denote separable locally convex spaces, and φ a linear mapping of X into Y.

Definition 1. φ is called **weakly continuous** if it is continuous for the weakened topologies $\sigma(X, X')$ in X and $\sigma(Y, Y')$ in Y, or, equivalently, if φ' exists. φ is called **weakly completely continuous** if it maps some neighborhood of zero in X into a weakly relatively bicompact set in Y.

We shall consider the following classes of spaces: class (I) of spaces X for which, for every Y, weak complete continuity of φ implies weak complete continuity of φ'; class (II) of spaces Y for which, for every X, weak complete continuity of φ implies weak complete continuity of φ'; class (III) of spaces X for which, for every quasicomplete Y, weak complete continuity of φ' implies weak complete continuity of φ.

Theorem 1. *In order that X be a space of class (I), it is necessary and sufficient that the topology of X coincide with the topology of uniform convergence on all absolutely convex $\sigma(X', X'')$-bicompact sets.*

Theorem 2. *Every metrizable Y is a space of class (II).*

Theorem 3. *Every quasibarrelled X possessing a fundamental sequence of bounded sets belongs to class (III).*

Theorem 4. *The following assertions are equivalent:*

a) X is a space of class (III).

b) For every Banach space Y, weak complete continuity of φ' implies weak complete continuity of φ.

c) For every absolutely convex $\sigma(X', X'')$-bicompact set K there exists a neighborhood of zero V in X such that K is weakly relatively bicompact in X'_V.

d) For every Banach space Y, every weakly continuous φ that maps bounded sets of X into weakly relatively bicompact sets in Y is weakly completely continuous.

Corollary 1. *If X is a space of class (III), then its topology majorizes the topology T of uniform convergence on all absolutely convex $\sigma(X', X'')$-bicompact sets.*

It follows from this that every quasinormable space X (ˆ3), whose topology majorizes T, belongs to class (III).

Corollary 2. *Every weakly continuous mapping of a semireflexive space of class (III) into a Banach space is weakly completely continuous.*

Theorem 5. The quotient space of a space of class (III), the inductive limit of a sequence of spaces of class (III), and the product of any family of spaces of class (III) are spaces of class (III).

Remark. A closed subspace of a space of class (III) need no longer be a space of this class. Thus, Grothendieck [3] constructs an example of a subspace X of types (F) and (M) possessing a quotient space isomorphic to the space l^1 of summable sequences. Let ψ be a homomorphism of X onto l^1. The map ψ carries bounded sets from X into relatively bicompact sets in l^1. At the same time $\psi(U)$ is a neighborhood of zero in l^1 for every neighborhood of zero U in X. Since l^1 is not reflexive, $\psi(U)$ cannot be a weakly relatively bicompact set in l^1. Therefore (Theorem 4) X does not belong to class (III). But X is a closed subspace of the product of a countable family of Banach spaces, which (Theorems 3 and 5) belongs to class (III).

Theorem 6. The strong conjugate of every space of class (III) is a space of class (II).

Theorem 7. If X is quasibarrelled and its strong conjugate X' is a space of class (II), then X belongs to class (III).

Definition 2 [4]. A locally convex space is called **countably centered** if, for every countable family of neighborhoods of zero (U_n), there exists a neighborhood of zero U absorbed by all the U_n.

Theorem 8. In the following two cases:
1) X is a countably centered space of class (III), Y is a space of type (F);
2) X is a bornologically closed space possessing a fundamental sequence of bounded sets, Y is metrizable,
every weakly continuous φ carrying bounded sets from X into weakly relatively bicompact sets in Y is weakly completely continuous.

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CITED LITERATURE

[1] V. Gantmakher, *Matem. sborn.*, **7** (49), 301 (1940).
[2] N. Bourbaki, *Topological Vector Spaces*, IL, 1959.
[3] A. Grothendieck, *Summa Brasil. Math.*, **3**, 57 (1954); Russian transl. in *Matematika*, **3**, 1958, p. 81.
[4] D. A. Raikov, *Uch. zap. Mosk. gos. ped. inst. im. V. I. Lenina*, No. 188, 171 (1962).

Note: Figure translations are in progress. See original paper for figures.