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1963

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Abstract

Full Text

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ON INFINITE-DIMENSIONAL UNITARY REPRESENTATIONS OF THE GROUP OF SECOND-ORDER MATRICES WITH ELEMENTS FROM A LOCALLY COMPACT FIELD*

(Presented by Academician I. G. Petrovskii, 4 I 1963)

1. Let K be a locally compact, nondiscrete field. Denote by G the group of all nonsingular second-order matrices with entries from the field K, and by G_0 the subgroup of matrices of the form

$$g_{a,b} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

Let $g \mapsto T(g)$ be an irreducible infinite-dimensional unitary representation of the group G in a Hilbert space H. We shall prove the following assertions.

Theorem 1. The restriction of the representation T to the subgroup G_0 is irreducible.

Theorem 2. In the decomposition of the representation T into irreducible representations of a maximal compact subgroup $C \subset G_0$, each component has finite multiplicity**.

Theorem 3. For any function $\varphi \in L^1(G)$, the operator

$$T(\varphi) = \int \varphi(g)T(g)\,dg$$

is completely continuous, and for all φ from some subspace dense in $L^1(G)$, the operator $T(\varphi)$ has finite rank.

Corollary. Every factor representation of the group G belongs to type I.

These results lead, in particular, to new examples of simple groups of type I, distinct from finite groups and Lie groups. Namely, such a group will be the subgroup $G_1 \subset G$ consisting of all matrices with determinant 1.

We note that for a connected field K (i.e., for the field of real or complex numbers) the theorems formulated above are well known. Therefore in what follows we assume the field K to be disconnected.

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2. We shall set forth the necessary facts about a disconnected locally compact nondiscrete field K (see ([3]), Ch. IV). The topology in the field K may be defined by means of a norm such that the set $O = \{x : |x| \leqslant 1\}$ is compact. The norm has the property

$$|x + y| \leqslant \max(|x|, |y|).$$

Therefore O is a ring. Let

$$O^* = \{x : x \in O,\ x^{-1} \in O\}.$$

It turns out that the multiplicative group K^* of the field K is the direct product of the compact group O^* and an infinite cyclic group with generator $\tau \in O \setminus O^*$. Consequently, every multiplicative character π is determined by a pair (θ, ν), where $\theta = \pi(\tau)$ is a complex number of absolute value 1, and ν is a character of the group O^*. Thus the group $\widehat{K}^*$, dual to the group K^*, is the product of the circle and the discrete group $\widehat{O}^*$, dual to the compact group O^*. In the group O^* there is a family of subgroups

$$O^*_n = \{1 + \tau^n x,\ x \in O\}.$$

We shall call the **order of the character** ν the least number n such that $\nu \equiv 1$ on O^*_n.

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* The representations of the corresponding unimodular group were constructed in ([1]). We use some notation from that work.

** This assertion was proved in ([2]) for certain special representations of the compact subgroup.

The additive group K^+ of the field K is dual to itself. We fix an additive character χ, equal to 1 on O and not identically equal to 1 on $\tau^{-1}O$. Every character of K^+ has the form $\chi_a(x) = \chi(ax)$, $a \in K$. We choose the Haar measure on K^+ so that the measure of O is equal to 1. There exists a norm in K such that the Haar measures dx on K^+ and d^*x on K^* are connected by the equality $dx = |x|\, d^*x$.

Define the Γ-function on $\widehat{K}^*$ by the formula

$$\Gamma(\pi) = \int_{K^*} \pi(x)\chi(x)\, d^*x.$$

It turns out that $\Gamma(\pi)$ is a generalized function on $\widehat{K}^*$. It is a continuous functional in the space of functions $\varphi(\pi) = \varphi(\theta, \nu)$, finite in ν and infinitely differentiable in θ.

Lemma 1. *Let $\Gamma(\pi) = \Gamma(\theta, \nu) = \sum \Gamma_k(\nu)\theta^k$ be the expansion of the Γ-function in a Fourier series in θ.*

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Then: 1) if the order of ν is equal to $m > 0$, then $\Gamma_k(\nu) = 0$ for all k except $k = -m$; $|\Gamma_{-m}(\nu)| = |\tau|^{m/2}$; 2) if the order of ν is equal to zero (i.e. $\nu \equiv 1$), then $\Gamma_k(\nu) = 0$ for $k < -1$, $\Gamma_{-1}(\nu) = -|\tau|$, $\Gamma_k(\nu) = 1 - |\tau|$ for $k \geq 0$.

3. We now proceed to the proof of Theorem 1.

Lemma 2. *The restriction of T to G_0 is a multiple of an irreducible representation.*

Proof. All irreducible unitary representations of the group G_0 can easily be described with the aid of Mackey' s theorem on induced representations (see, for example, (4), § 5). All of them, except one, are one-dimensional and have the form $V(g_{a,b}) = \pi(a)$. The unique infinite-dimensional irreducible representation is realized in the space $L^2(K^*, d^*x)$ and has the form

$$U(g_{a,b})\varphi(x) = \chi(bx)\varphi(ax).$$

The restriction of T to G_0, like any unitary representation of G_0, can be realized in the form of a direct integral of irreducible representations. We shall show that the one-dimensional representations of G_0 cannot constitute a set of positive measure in this decomposition. Indeed, otherwise in the space H there would be a vector ξ invariant with respect to the subgroup $N \subset G_0$ consisting of the matrices $g_{1,b}$. Then the function $F_\xi(g) = (T(g)\xi, \xi)$ would be a bounded continuous positive definite function on G, constant on double cosets of adjacency modulo N. Simple calculations show that such a function must have the form $F_\xi(g) = \pi(\det g)(\xi, \xi)$, where π is a fixed character on K^*. Hence $T(g)\xi = \pi(\det g)\xi$, which contradicts the irreducibility of T.

Thus, in the decomposition of T into a direct integral of irreducible representations of G_0, all components are equivalent to the representation U. In this case the direct integral may be replaced by a discrete sum of representations equivalent to U. The lemma is proved.

It will be convenient for us to pass to another realization of the representation U, considering, instead of functions on K^*, their Fourier transforms on the dual group $\widehat{K}^*$.

In this realization the representation operators have the form:

$$U(g_{a,b})\varphi(\pi_1) = \int \frac{\pi_1(b)}{\pi_2(b)}\Gamma\left(\frac{\pi_2}{\pi_1}\right)\pi_2(a)\varphi(\pi_2)\,d\pi_2 \quad \text{for } b \neq 0,$$

$$U(g_{a,0})\varphi(\pi) = \pi(a)\varphi(\pi). \tag{1}$$

It follows from Lemma 2 that the restriction of T to G_0 is given by the same formulas, only instead of ordinary functions one must consider vector-valued functions with values in some Hilbert space L.

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Moreover, for g from the center of G, the operators $T(g)$ commute with all operators of the representation and therefore are multiples of the identity operator. Hence it follows that if

$$d_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

then $T(d_\lambda) = \pi_0(\lambda)E$, where π_0 is a fixed

character on K^*. Denote by s the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. From the identity $sg_a, 0s^{-1} = g_{a^{-1}}, 0d_a$ it follows that the operator $T(s)$ has the form

$$T(s)\varphi(\pi) = s(\pi)\varphi(\pi_0\pi^{-1}).$$

where $s(\pi)$ is a function on $\widehat{K}^*$, whose values are unitary operators in L.

Let us now note that the irreducibility of T implies the irreducibility of the family of operators $s(\pi)$ in L. Indeed, if L_1 is a subspace in L, invariant with respect to all (or even almost all) $s(\pi)$, then the subspace $H_1 \subset H$, consisting of vector-functions with values in L_1, is invariant with respect to $T(g)$, $g \in G_0$, and with respect to $T(s)$. But the subgroup G_0 and the element s generate the whole group G. Therefore H_1 is invariant with respect to all $T(g)$, which contradicts the irreducibility of T.

To prove Theorem 1 it remains only to show that the operators $s(\pi)$ commute with one another. Then the family $s(\pi)$ will be irreducible only in the case when L is one-dimensional and, consequently, the restriction of T to G_0 coincides with U. The identity $sg_{1,1}s = g_{1,-1}sg_{1,-1}$ leads to the following condition on $s(\pi)$:

$$s(\pi_1)\Gamma(\pi_1\pi_2\pi_0^{-1})s(\pi_2) = \pi_1(-1)\pi_2(-1)\int \Gamma(\pi\pi_1^{-1})s(\pi)\Gamma(\pi\pi_2^{-1})\,d\pi, \qquad (2)$$

from which the commutativity of $s(\pi_1)$ and $s(\pi_2)$ for almost all pairs (π_1, π_2) follows immediately. Theorem 1 is proved.

4. Thus the space L is in fact one-dimensional and $s(\pi)$ is an ordinary function taking complex values of modulus 1. The problem of describing all representations of the group G is reduced to the problem of finding functions $s(\pi)$ satisfying condition (2).

We turn to the proof of Theorem 2. Condition (2), in terms of the Fourier coefficients of the function $s(\pi)$, has the form

$$\sum_m s_{k+m}(\nu_1)\Gamma_{-m}(\nu_1\nu_2\nu_0^{-1})s_{l+m}(\nu_2) =$$

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$$= \nu_1(-1)\nu_2(-1)\sum_{\nu}\Gamma_{-k}(\nu\nu_1^{-1})s_{k+l}(\nu)\Gamma_{-l}(\nu\nu_2^{-1}). \quad (3)$$

We shall investigate this condition, taking into account the results of Lemma 1.

First, considering (3) for fixed ν_1, ν_2, l and sufficiently large positive k, we see that the right-hand side is zero, while the sum on the left-hand side, for $\nu_1, \nu_2 \neq \nu_0$, reduces to one term in which m is equal to the order of $\nu_1\nu_2\nu_0^{-1}$. Hence it is easy to conclude that, for each ν, the coefficients $s_k(\nu)$ vanish for sufficiently large positive k.

Second, if $k \leq 0,\ l \leq 0,\ \nu_1 \neq \nu_2,\ \nu_1\nu_2 \neq \nu_0$, then from (3) follows the equality $s_{k+m}(\nu_1)s_{l+m}(\nu_2) = 0$, where m is the order of $\nu_1\nu_2\nu_0^{-1}$. Suppose that for some ν_1 and some $n \leq 0$ the coefficient $s_n(\nu_1)$ is nonzero. Then, putting $k = n - m$, we obtain that for all ν_2 distinct from ν_1 and $\nu_0\nu_1^{-1}$, the coefficients $s_{l+m}(\nu_2)$ are zero for $l \leq 0$. We have thus proved that, for all ν, except possibly one pair $\nu_1, \nu_0\nu_1^{-1}$, among the coefficients $s_k(\nu)$ only a finite number are nonzero. Finally, for the "exceptional" characters ν_1 and $\nu_0\nu_1^{-1}$, from (3) one can obtain recurrence relations between the $s_k(\nu)$, from which it follows that $|s_k(\nu)|$ decreases as $k \to -\infty$ like a geometric progression.

From all that has been said it follows

Lemma 3. The function $s(\pi) = s(\theta, \nu)$ is infinitely differentiable in θ for each ν.*

We are now in a position to prove Theorem 2. Let us first note that all maximal compact subgroups in the group G are conjugate to the group C,

* Condition (3) makes it possible to obtain more precise information about $s(\pi)$, which we shall not need now.

consisting of those matrices g for which the matrix elements of g and g^{-1} belong to O. In the group C there is a family of normal subgroups C_n, consisting of matrices congruent to the identity matrix modulo $\tau^n O$. Every irreducible representation of the group C is trivial on C_n for sufficiently large n. Denote by H_n the subspace in H consisting of vectors invariant with respect to the operators $T(g)$, $g \in C_n$. Theorem 2 is equivalent to the assertion that all the spaces H_n are finite-dimensional.

Let us first find the space H_n^0, consisting of vectors invariant with respect to $T(g)$, $g \in C_n \cap G_0$. This is easiest to do using the original realization of the representation U. We shall formulate only the final result.

Lemma 4. *The space H_n^0 consists of functions*

$$\varphi(\pi) = \sum \varphi_k(\nu)\theta^k,$$

satisfying the condition: $\varphi_k(\nu) = 0$ if $(\operatorname{ord}\nu) \geq n$ or $k < -n$.

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Since $sC_{ns}^{-1} = C_n$, H_n is invariant with respect to $T(s)$. Therefore, if $\varphi(\pi) \in H_n$, then the functions $\varphi(\pi)$ and

$$T(s)\varphi(\pi) = s(\pi)\varphi(\pi_0\pi^{-1})$$

simultaneously satisfy the conditions of Lemma 4. The finite-dimensionality of the space H_n now follows from Lemma 3 and the following proposition:

Lemma 5. *Let $s(\theta)$ be an infinitely differentiable function on the unit circle Θ, and let $|s(\theta)| \equiv 1$. In the space $L^2(\Theta)$ there exists only a finite number of linearly independent functions $\varphi(\theta)$ satisfying the conditions: 1) the function $\varphi(\theta)$ is orthogonal to θ^k for $k < -n$; 2) the function $s(\theta)\varphi(\theta)$ is orthogonal to θ^k for $k > n$.*

5. Theorem 3, as is known, is derived from Theorem 2 by standard arguments. One can indicate the following direct route for proving this theorem. Let $\widetilde{\varphi}_{k,\nu}$ be a generalized function on G, given by the formula

$$(\widetilde{\varphi}_{k,\nu}, f) = \int f(g_{a,b})\,\overline{\nu(a)}\,d^*a\,db,$$

where the integral is taken over the set

$$\{(a,b) : a \in O^*,\ b \in \tau^{-k}O\}.$$

Put

$$\varphi_{k,\nu} = \widetilde{\varphi}_{k,\nu} - \widetilde{\varphi}_{k-1,\nu}.$$

It is not difficult to verify that the operator $U(\varphi_{k,\nu})$ is the projector onto the one-dimensional subspace in $L^2(K^*)$ generated by the function

$$e_{k,\nu}(x) = \begin{cases} \nu(\tau^{-k}x), & \text{if } x \in \tau^k O^*, \\ 0, & \text{if } x \notin \tau^k O^*. \end{cases}$$

Obviously, the collection of functions $e_{k,\nu}(x)$ forms an orthogonal basis in $L^2(K^*)$.

Let now M be the collection of all functions $\varphi \in L^1(G)$ for which the operator $T(\varphi)$ has finite rank. It is clear that M is a two-sided ideal in $L^1(G)$, containing all functions of the form $\varphi_{k,\nu} * f$, $f \in L^1(G)$. If $u \in L^\infty(G)$ is a functional on $L^1(G)$ equal to zero on M, then the function u and all its translates have the property

$$u * \varphi_{k,\nu} = 0.$$

Hence $u = \text{const}$, and the closure of M contains all functions from $L^1(G)$ whose integral is equal to zero. From Mautner' s results [5] it follows that in M there are functions with nonzero integral. Hence

$$\overline{M} = L^1(G),$$

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and Theorem 3 is proved.

The author expresses his gratitude to I. M. Gel' fand and M. I. Graev for discussing the present note.

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Received
2 I 1963

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