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Abstract

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MATHEMATICS

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EXTREMUM PROBLEMS IN THE PRESENCE OF CONSTRAINTS

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This note gives a new approach to problems of the calculus of variations. It turns out that also "nonclassical" problems, such as problems of optimal regulation, can be studied by the same methods by which "classical" problems are studied. We consider those necessary conditions for an extremum which can be obtained by means of the first variation. In the "classical" case these conditions naturally lead to the Euler equation.

We give a brief description of the proposed method. Suppose that in some complete normed space W a functional $F(w)$ is given. It is required to find the minimum of $F(w)$ under the condition that a certain finite number of constraints are imposed on w. We shall distinguish constraints of inequality type and constraints of equality type. Each constraint of inequality type is the closure of a set open in W. We shall also assume that constraints of equality type single out a certain "smooth" manifold in the space W. It is clear that every point w satisfying all the constraints is contained in the intersection of these sets.

Let w^0 be a point of minimum of the functional F, satisfying all the constraints. We shall say that $\bar{w} \in W$ is a forbidden variation if

$$\frac{d}{d\varepsilon} F(w^0 + \varepsilon \bar{w}) \Big|_{\varepsilon=0} = F'(w^0, \bar{w}) < 0.$$

We shall always assume that $F'(w^0, \bar{w})$ depends continuously on $\bar{w}$.

Now consider one of the constraints of inequality type. A variation $\bar{w}$ will be called admissible with respect to the given constraint if $w^0 + \varepsilon \bar{w}$ satisfies it for all sufficiently small ε. Similarly, a variation $\bar{w}$ is considered admissible with respect to constraints of equality type if the line $w^0 + \bar{w}$ is tangent to the distinguished manifold at the point w^0. It is evident that both the forbidden variations and the variations admissible with respect to the various constraints form conic sets in the space W. In this note we assume that:

1. The forbidden variations, as well as the variations admissible with respect to constraints of inequality type, form convex cones with interior points.
2. The variations admissible with respect to constraints of equality type form some subspace L. Under quite general assumptions on the functional F, one can assert that a necessary condition for a minimum in W^0 is the absence of common points among the open parts of all admissible cones, the cone of forbidden variations, and the subspace L. However, in order to obtain a substantive criterion it is necessary to formulate this condition in terms of linear functionals; the latter is clear at least from the circumstance that all constraints "in the small" are written with the aid of linear inequalities, and the information about these inequalities must enter into the necessary conditions for a minimum.

Thus the following problem arises: given a finite system of open convex cones $\Omega_1, \dots, \Omega_s$ and a subspace L. It is required to find a necessary and sufficient condition in terms of dual spaces for the intersection $\Omega_1 \dots \Omega_s \cdot L = 0$. This

the problem has the following solution. Let Ω_i^* denote the set of linear functionals nonnegative on Ω_i, and let L^* denote the set of linear functionals vanishing on L.

Theorem. In order that $\Omega_1 \dots \Omega_s \cdot L = 0$, it is necessary and sufficient that there exist linear functionals $\omega_1, \dots, \omega_s, l$, $\omega_i \in \Omega_i^*, l \in L^*$, not all equal to zero simultaneously, and such that

$$\omega_1 + \dots + \omega_s + l = 0. \tag{1}$$

We note that condition (1) plays the same role as the Euler equation and, in the "classical" case, coincides with it.

Examples.

1. Euler equation. Let

$$I(x) = \int_0^1 f(t, x, \dot{x})\, dt.$$

It is required to find conditions for a minimum of the functional I, if $x(0) = x_0$ and $x(1) = x_1$. We reformulate the problem as follows: in the space of pairs of continuous functions $x(t), u(t)$, $0 \le t \le 1$, a functional is given,

$$I(x, u) = \int_0^1 f(t, x, u)\, dt.$$

Find necessary conditions for a minimum of $I(x,u)$ under the constraints

$$x = x_0 + \int_1^t u\,dt, \qquad x(1) = x_1.$$

Let $x^0(t), u^0(t)$ be a minimum point of the functional I, satisfying all the constraints. The cone of forbidden variations Ω consists of those $\bar{x}, \bar{u}$ for which

$$I'(x^0, u^0; \bar{x}, \bar{u}) = \int_0^1 \left(\frac{\partial f}{\partial x}\bar{x} + \frac{\partial f}{\partial u}\bar{u}\right) dt < 0.$$

The equality-type constraints lead to the subspace

$$L\bar{x} = \int_0^t \bar{u}\,dt, \qquad \bar{x}(1) = 0.$$

Since I' is a linear functional of $\bar{x}, \bar{u}$, we have

$$\omega(\bar{x}, \bar{u}) = -\alpha I',$$

where ω is an arbitrary element of Ω^*. It is known that any linear functional in the space of continuous functions can be written in the form of a Stieltjes integral:

$$l(\varphi) = \int_0^1 \varphi(t)\,d\mu(t),$$

where $\mu(t)$ is some measure. Using this fact, any linear functional $l \in L^*$ can be represented in the form

$$\int_0^1 \left(\bar{x} - \int_0^t \bar{u}\,dt\right) d\mu(t) + c\bar{x}(1) = l(\bar{x}, \bar{u}).$$

Condition (1) therefore leads to the equality

$$-\alpha \int \left(\frac{\partial f}{\partial x}\bar{x} + \frac{\partial f}{\partial u}\bar{u}\right) dt + \int \left(\bar{x} - \int_0^t \bar{u}\,dt\right) d\mu(t) + c\bar{x}(1) = 0$$

and to the inequality $\alpha \geq 0$. We emphasize that the equality must hold identically in $\bar{x}, \bar{u}$. Equating the terms with $\bar{x}$ and $\bar{u}$ separately, we obtain:

$$\alpha \frac{\partial f}{\partial u} - \alpha \int_1^t \frac{\partial f}{\partial x}\, dt - c = 0.$$

2. L. S. Pontryagin' s maximum principle.

1) On solutions of the system

$$\frac{\partial x}{\partial t} = f(x, u) \tag{2}$$

the functional is given by

$$I(x, u) = \int_{t_0}^{t_1} \Phi(x, u)\, dt.$$

Among all $x(t)$, $u(t)$ satisfying (2) and such that $x(t_0) = x_0$, $x(t_1) = x_1$, find those on which I assumes its least value. The instants t_0 and t_1 are not fixed. The values $u(t)$ belong to some set D in E^r. f and Φ are continuously differentiable with respect to x and continuous with respect to u. In this problem the issue is necessary conditions for a strong extremum, i.e. the following notion of closeness is used. A sequence $u_n(t)$ tends to $u(t)$ if $|u_n(t)|$ are uniformly bounded and converge to $u(t)$ in measure.

First of all, let us note that carrying out direct variation in such a topology is very inconvenient because of its "nondifferentiability." For example, let $u(t, \varepsilon) = e^{-t^2/\varepsilon^2}$, $0 \le t \le 1$. Obviously, $u(t, \varepsilon) \to 0$ as $\varepsilon \to 0$ in the given topology. However, $u(t, \varepsilon)$ cannot be represented in the form $u(t, \varepsilon) = \varepsilon \bar{u}(t) + o(\varepsilon)$. In view of the latter circumstance, the problem under consideration cannot be studied by the classical methods of the calculus of variations.

Therefore the following device turns out to be natural. Any function $u(t)$ can be written in the form $u(\tau_v(t))$, where $u(\tau)$ is some function, and τ and t are connected by the relation $dt/d\tau = v(\tau)$ $(v \ge 0)$. Obviously, $u(\tau)$ is uniquely determined where $v(\tau) \ne 0$, and may be prescribed arbitrarily where $v(\tau) = 0$.

Consider $u(t, \varepsilon) = u(\tau_{v+\varepsilon\bar{v}}(t))$, $v + \varepsilon\bar{v} \ge 0$ for $\varepsilon \le \varepsilon_0$. Then, as $\varepsilon \to 0$, $u(t, \varepsilon) - u(t)$ tends to zero in the sense indicated above. It is important that $v(\tau, \varepsilon) = v + \varepsilon\bar{v} \to v(\tau)$ in a manner differentiable with respect to ε, whereas $u(t, \varepsilon) \to u(t)$, as a rule, in a nondifferentiable manner. It follows from this that, in order to preserve the apparatus of the calculus of variations, one must vary $v(\tau)$, and here there arises the natural constraint $v(\tau) \ge 0$.

2) **An equivalent formulation of the problem in item 1), convenient for variation.** Let $u^0(t)$ be an optimal control; t_0, t_1 the time of its action, and $x^0(t)$ a solution of system (2). Let, further, $v^0(\tau)$ be some nonnegative function on $(0, 1)$

$$\left(\int_0^1 v^0(\tau)\,d\tau = t_1 - t_0\right).$$

Set $dt/d\tau = v_0(\tau)$. Further take $u^0(\tau) = u^0(t(\tau))$, $x^0(\tau) = x^0(t(\tau))$. Finally, let $\tilde{u}^0(\tau)$ coincide with $u^0(\tau)$ where $v^0(\tau) \neq 0$ and be specified in some way on the set of zeros of $v^0(\tau)$.

The problem is formulated as follows: among all $x(\tau), v(\tau)$ satisfying the system

$$dx/d\tau = vf(x, \tilde{u}^0),$$

find those on which the functional

$$I(x, v) = \int_0^1 v\Phi(x, \tilde{u}^0)\,d\tau$$

attains a minimum. The following constraints are imposed on x and v: $x(0) = x_0$, $x(1) = x_1$ and $v \geq 0$. It is clear that $v^0(\tau), x^0(\tau)$ is a solution of this problem. Investigation of the problem according to the method set forth at the beginning leads to L. S. Pontryagin' s maximum principle.

3) **The minimax problem.** In the same formulation as in item 1), it is required to minimize the functional

$$I(x, u) = \max_{t_0 \leq t \leq t_1} \Phi(x).$$

Transforming the problem analogously to item 2) and then studying it according to the proposed method, we obtain: let a be an arbitrary solution of the system

$$-\frac{da}{dt} = \left(\frac{\partial f}{\partial x}\right)^* a, \tag{3}$$

where $\partial f/\partial x$ is taken at the optimal $x(t), u(t)$, and $(\partial f/\partial x)^*$ is the matrix transposed to it. $\tilde{a}_{t'}(t)$ satisfies system (3) for $t \leq t'$ and is equal to zero for $t > t'$; $\tilde{a}_{t'}(t') = \Phi'(x(t'))$, μ is a certain measure given on (t_0, t_1); $\psi_\mu(t) = \int \tilde{a}_{t'}\,d\mu(t')$; M is the set of points t at which $\Phi(x(t))$ attains its greatest value.

Theorem (maximum principle). Let $u^0(t)$ be an optimal control and $x^0(t)$ the corresponding solution. There exist $a(t)$ and a nonnegative measure μ, concentrated on M, not both identically zero, such that:

$$(a(t) + \psi_\mu(t),\ f(x^0(t), u)) \geq 0, \qquad (a, f) = \sum_1^n a_i f_i$$

for all $u \in D$;

$$(a(t) + \psi_\mu(t),\ f(x^0(t), u^0(t))) = 0$$

for all (or almost all, depending on the class of admissible $u(t)$) values $t_0 \le t \le t_1$.

3. S. N. Bernstein' s inequality. In the space of polynomials of degree n, the functional $F(p) = \max_{-\infty<x<\infty} |p'/s'|$ is given, where s is a polynomial of degree n with roots lying below the real axis. It is required to find $\max F(p)$ under the condition $|p/s| \le 1,\ -\infty < x < \infty$. Let p_0 be the solution of the problem. Variation of the inequality-type constraint leads to the convex cone Ω of those $\bar{p}$ for which $\operatorname{Re} \bar{p}/p_0 \le 0,\ x \in M$, where M is the set of points of the real axis on which $|p_0/s| = 1$.

Direct computation gives $F'(p_0, \bar{p}) = F(p) \max_{x \in M_1} \operatorname{Re} \bar{p}_1'/p_0'$, where on M_1 the ratio p_0'/s' attains its maximum. The cone K of forbidden variations consists of the $\bar{p}$ for which $F' > 0$. In the case under consideration the cone of forbidden variations is not convex. Therefore condition (1) cannot be applied directly.

The necessary condition for an extremum in this case can be formulated as follows: whatever half-space $\pi \subset K$ is taken, one has $\pi \cdot \Omega = 0$.

If we put $K = \bigcup \pi$, then $-(cK)^* = \bigcup \pi^*$. Therefore from condition (1) it follows: for any functional $\omega_1 \in (cK)^*$ there is a functional $\omega \in \Omega^*$ such that

$$\omega_1 = \omega. \tag{4}$$

Using the general form of a linear functional in the space of continuous functions, we conclude that for any nonnegative measure μ_1 concentrated on M_1 there is such a nonnegative measure μ, concentrated on M, that

$$\int \frac{\bar{p}'}{p_0'}\, d\mu_1 = \int \frac{\bar{p}}{p_0}\, d\mu$$

for any polynomial $\bar{p}$ of degree n. Analysis of the latter leads to the inequality $|p_0'| \le |s'|$.

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Note: Figure translations are in progress. See original paper for figures.