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Abstract

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On the Question of the Flow of a Nonequilibrium Gas

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The paper considers gas flows with relaxation of a relatively low-energy degree of freedom or reaction, called flows of a weakly relaxing gas (A), as well as flows close to equilibrium (B); an explicit form is obtained for the structure term for a shock wave in case (A) for relaxation of vibrational degrees of freedom; using vibrational relaxation as an example, in case (B) the question is studied of obtaining systems of differential equations from a system of integro-differential equations when the mean free path is small in comparison with the characteristic length.

§ 1. Gas particles, as is well known, possess translational, vibrational, rotational, etc. degrees of freedom; at high temperatures dissociation, ionization, and also various chemical reactions occur. Since most often the corresponding relaxation times differ in order of magnitude and, if one assumes relatively rapid relaxation within each degree of freedom, we arrive at the generally accepted scheme for considering the phenomenon of relaxation (see, for example, ([1])), which we shall use below. This scheme proceeds from the existence of thermodynamic equilibrium within each degree of freedom, characterized by a temperature T_i or concentration α_i; the study of the relaxation phenomenon is then reduced to the question of the redistribution of energy among the excited degrees of freedom.

In considering gas flows with relaxation, the following additional characteristic parameters appear, beyond those usually considered in hydrodynamics:

1) $\omega_i = L/v\tau_i$ (L is the characteristic length; τ_i is the characteristic relaxation time of the i-th degree of freedom; v is the characteristic particle velocity; $i = 1, 2, \dots, N$). The parameters ω_i characterize the degree to which the flow characteristics differ from their equilibrium values; thus, for all $\omega_i \gg 1$ the gas flow will be close to equilibrium; conversely, for $\omega_i \ll 1$ the characteristics of the i-th degree of freedom will be almost frozen.

2) $W_i = E_i/h_0$ (E_i is the characteristic value of the internal energy of the i-th degree of freedom; h_0 is the total enthalpy; $i = 1, 2, \dots, N$). The parameter W_i characterizes the relative energy capacity of the i-th degree of freedom.

In what follows, bearing in mind application to external problems of gas dynamics, we assume that $h \sim h_0$.

In the case of small values of the parameter W_i,* the flow characteristics p_n $(n = 1, 2, \dots, l)$ can be determined by means of a series

$$p_n = \sum_{k=0}^{\infty} c_{nk} W_i^k.$$

As the zeroth term one takes the characteristics of a gas with $W_i = 0$, while the relaxation equation itself (the equation for T_i or α_i) is integrated in this approximation, since it contains known functions of the coordinates, associated –

* Which particular argument (T or T_i) is adopted in calculating W_i is immaterial, since from the smallness of $W_i(T)$ follows the smallness of $W_i(T_i)$.

⋯with the flow, $W_i = 0$; the first term of the series will be the solution of the system of linear equations for the gas-dynamic characteristics of the flow and the determination of the correction to the temperature distribution T_i, starting from the relaxation equation, etc.

If all $W_i \ll 1$, then we shall call the flow a flow of a weakly relaxing medium. In particular, the quantity $W_i \ll 1$ if the equilibrium temperature of the gas does not differ greatly from that at which excitation of the i-th degree of freedom occurs, which may take place in the calculation of flows with dissociation or ionization at an early stage.

Let us now turn to the scheme of vibrational relaxation. The equations for stationary motions of a gas in this case have, as is known (see, for example, ([2])), the form:

$$\operatorname{div} \rho \mathbf{v} = 0; \qquad (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p; \qquad v^2/2 + c_{pT} + E_i = h_0; \qquad p = \rho R T,$$

$$\frac{dE_i}{dt} = \frac{E_i(T) - E_i}{\tau} \qquad E_i(T_i) = \frac{R\theta_v}{e^{\theta_v/T_i} - 1}; \tag{1}$$

ρ is the density; $\mathbf{v}$ is the velocity vector; p is the pressure; T is the temperature of the translational degrees of freedom; h_0, c_p, R, θ_v are constants.

If we now assume that the medium is weakly relaxing, the corrections to the gas-dynamic elements due to allowance for relaxation (denoted by a prime index) in the first approximation satisfy the following system of equations:

Fig. 1

Figure 1: Fig. 1

$$\operatorname{div}\rho_0\mathbf{v}' + \operatorname{div}\rho'\mathbf{v}_0 = 0; \qquad (\mathbf{v}_0\nabla)\mathbf{v}' + (\mathbf{v}'\nabla)\mathbf{v}_0 = -\frac{1}{\rho_0}\nabla p' + \frac{\rho'}{\rho_0^2}\nabla p_0;$$

$$\mathbf{v}_0\mathbf{v}' + c'_{pT} + E'_i = 0; \qquad p' = \rho' RT_0 + \rho_0 RT'; \tag{2}$$

$$E'_i = \exp\left[-\int_0^s \frac{dt}{\tau_0 v_0}\right]\int_0^s \exp\left[\int_0^s \frac{ds}{\tau_0 v_0}\right]\cdot E_i(T_0)\frac{ds}{\tau_0 v_0}.$$

The integrals in the expression for E'_i are evaluated along the streamlines of the flow in the zeroth approximation. The system of equations (2) takes an especially simple form in the case when the flow parameters with subscript zero have constant values, as will be the case, for example, for the flow of a weakly relaxing gas over a wedge by a supersonic stream.

In Fig. 1 a plot is given of the quantity

$$\frac{\chi}{\chi-1}\frac{E_i(T)}{c_{pT}} = f\left(\frac{\theta_v}{T}\right)$$

(see the last equation of system (1)); as is seen, in the case of vibrational relaxation the quantity W_i is small over the entire temperature range—of order $(\chi-1)/\chi$, while in the case T_0 = const the maximum is $(\chi-1)/\chi$. Thus, the method of successive approximations described above for vibrational relaxation in the case of large temperatures (when $\theta_v/T \ll 1$) is transformed into the method of expanding the solution in a series in powers of the parameter $(\chi-1)/\chi$.

Fig. 1

§ 2. Let us consider, as an example of the application of the small-parameter method set forth above, the structure of a plane shock wave in the case of vibrational relaxation.

Thus, suppose that $E_i/c_{pT}0 \ll 1$. Then the structural term due to vibrational relaxation can be obtained in the form of a small variation of the ordinary shock wave, whose thickness is of the order of the relaxation length of the translational degrees of freedom, i.e., according to our assumption, equal to zero.

The equations describing the shock wave have the form:

$$\rho u = C_1; \qquad p + \rho u^2 = C_2; \qquad u^2/2 + c_{pT} + E_i = h_0. \tag{3}$$

If we assume that the shock wave corresponding to the translational degrees of freedom is located at the point $x = 0$, and if $E_i\big|_{x=0} = 0$, then the expression for $E_i'(x)$ will be:

$$E_i'(x) = E_i(T_0)\{1 - \exp[-x/u_0\tau_0]\} \tag{4}$$

(the index 0 denotes quantities in the shock wave corresponding to the translational degrees of freedom).

Linearizing the system of equations (3) and solving it with respect to the perturbations, we obtain:

$$T' = -\frac{M_0^2 - 1/\varkappa}{M_0^2 - 1}\frac{E_i'(x)}{c_v}; \qquad p' = -\frac{(\varkappa-1)M_0^2\rho_0}{M_0^2 - 1}E_i'(x); \tag{5}$$

$$\rho' = \frac{p'}{u_0^2}; \qquad u' = \frac{M_0^2(\varkappa-1)}{M_0^2 - 1}\frac{E_i(x)}{u_0}$$

(M_0 is the Mach number after the ideal shock wave).

Thus, owing to excitation of the vibrational degrees of freedom after the ideal shock wave, a deceleration of the flow is observed, an increase in pressure and, if $M_0^2 < 1/\varkappa$, a decrease in the static temperature.

In the case of large values of T/θ_v, if we also assume that $\varkappa = 1.4$ and $M_0 = (\varkappa - 1)/2\varkappa$ (which corresponds to an infinitely large value of the Mach number before the shock wave), then expressions (4), as $x \to \infty$, give:

$$T' \approx -0.2667T_0; \qquad p' \approx 0.0667\,p_0; \qquad u' \approx -0.3333\,u_0; \qquad \rho' \approx 0.3333\,\rho_0. \tag{6}$$

Calculating the second approximation for the conditions under which expressions (6) were obtained, we find:

$$T'' \approx 0.0629\,T_0; \qquad p'' \approx -0.0289\,p_0; \qquad u'' \approx 0.1143\,u_0; \qquad \rho'' \approx -0.0032\,\rho_0. \tag{7}$$

Thus, the first approximation (5) describes the structure of the shock wave for the case of vibrational relaxation with acceptable accuracy over the entire temperature range.

§ 3. Flows of an almost equilibrium gas ($\omega_i \gg 1$), chiefly their computational aspect, have already been investigated (see, for example, ([3],[4])). Below we shall dwell on the qualitative aspect of the question of these flows.

First of all, let us note that the system of equations (1) can be transformed into an integro-differential system of equations with respect to the quantities determining the equilibrium flow. Indeed, formally integrating the relaxation equation, we obtain

$$E_i(T_i) = \exp\left[-\int_0^s \frac{ds}{v\tau}\right] \int_0^s \left\{\exp\left[\int_0^s \frac{ds}{u\tau}\right] \cdot E_i(T)\right\} \frac{ds}{v\tau}, \tag{8}$$

i.e., from the first four equations of system (1), with the help of expression (8), the temperature of the vibrational degrees of freedom T_i is eliminated.

As a result we obtain a system of integro-differential equations describing the dissipative process associated with vibrational relaxation. Mathematically, it is in many respects analogous to the processes of viscous dissipation or radiative dissipation. Naturally, when the "mean free path" becomes small in comparison with the characteristic dimension (i.e., $\omega_i \gg 1$), the obtained system of integro-differential equations admits a transition to a differential one. Let us find the system of differential equations. For

for this purpose it is natural to expand the expressions $1/v\tau$ and $E_i(T)$ in formula (8) in series in the neighborhood of some point S_M and to compute the corresponding integrals; as a result we obtain an expression for $E_i(T_i)$ in the form of a series arranged in powers of $1/\omega_i$. In deriving this series it will be seen that it is valid only for interior points of the region, located at a distance from the boundary at least of the order of several relaxation lengths ($l = v\tau$). Formally, the desired series can also be obtained in another way, by solving the relaxation equation for $E_i(T_i)$ and substituting the resulting expression n times into itself; in the end we shall have

$$E_i(T_i) = E_i(T) - \frac{dE_i(T)}{dt}\frac{1}{\omega_i} + \cdots + (-1)^n \frac{d_i^{nE}(T)}{dt^n}\frac{1}{\omega_i^n} + \cdots \tag{9}$$

$$\left(\omega_i \bar{t} = \int_0^s \frac{ds}{v\tau}\right).$$

The third term of the expansion (9) is an analogue of the Burnett terms in the aerodynamics of a rarefied gas.

Thus, the system of equations of motion for an almost equilibrium medium with vibrational relaxation can, in the principal term, be written in the form

$$\operatorname{div}\rho\mathbf{v} = 0; \qquad (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p; \qquad \frac{v^2}{2} + h - \tau(\mathbf{v}\nabla)E_i(T) = h_0; \qquad p = \rho RT$$

$$(h = c_{pT} + E_i(T)). \tag{10}$$

Instead of two parameters characterizing the influence of relaxation, we now have one, $\overline{W} = E_i \tau v / h_0 L$.

At first glance there arises the problem of finding additional boundary conditions, since the order of the system of equations (10) is increased in comparison with the order of the system of equations for a thermodynamically equilibrium gas. However, in accordance with the derivation of the system of equations (10), there are no additional boundary conditions, since the equations are considered for interior points of the flow region, where relaxation disturbances from the boundary decay, while the additional term in the energy equation due to relaxation must be interpreted as a distributed heat source. In other words, by virtue of the assumption $\omega_i \gg 1$, we obtain that $\overline{W} \ll 1$, and changes in the characteristics due to relaxation may be regarded as small additions to the corresponding characteristics of equilibrium flow, i.e., the additional term in the energy equation is a prescribed function of the parameters of the equilibrium flow. The same is true of all terms of the series (9), since what is involved is in fact only a method of successive approximations.

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Note: Figure translations are in progress. See original paper for figures.