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Abstract

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ON SOME PROPERTIES OF THE DISCRETE SPECTRUM OF THE RADIAL SCHRÖDINGER EQUATION

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Let us consider the Schrödinger equation

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left[2E+\frac{2A}{r}-\frac{l(l+1)}{r^2}-V(r)\right]R=0,\qquad R(0)=0. \tag{1}$$

If

$$\int_0^\delta r|V(r)|\,dr<\infty \quad\text{and}\quad \int_N^\infty |V(r)|\,dr<\infty$$

for arbitrary $\delta>0,\ N>0$, then, as is known, the limit exists

$$\lim_{r\to\infty}\left[R(r)e^{-\varepsilon r}r^{A/\varepsilon+1}\right]=C(\varepsilon,l),\qquad \text{where } \varepsilon=\sqrt{-2E}\ (\operatorname{Re}\varepsilon>0); \tag{2}$$

$R(r)$ is the solution of equation (1), regular at zero, normalized so that

$$\lim_{r\to 0}\frac{R(r)}{r^l}=1. \tag{3}$$

Here the function $C(\varepsilon,l)$ is analytic in the domain $\operatorname{Re}\varepsilon>0$, $\operatorname{Re}l>-1/2$.

1. We shall study in detail the function $C(\varepsilon,l)$ for the case when

$$V(r)=\sum_{p=-(n-1)}^{n-1}\frac{B_p}{r^{3/2+p/2n}}. \tag{4}$$

In this case the solution $R(r)$, regular at zero, has the form

$$R(r)=e^{-\varepsilon r}r^l\sum_{\nu=0}^{\infty}a_\nu r^{\nu/2n}, \tag{5}$$

where $a_0=1$.

Introduce the following notation:

$$s = l - \frac{A}{\varepsilon}; \qquad a_{2nm+k} = a(m,k), \qquad 0 \le k < 2n;$$

$$b(m,k) = (2\varepsilon)^m \prod_{q=1}^{m} \frac{q + s + k/2n}{(q + k/2n)(q + 2l + 1 + k/2n)}, \qquad 0 \le k < 2n; \quad m = 1, 2, \dots;$$

$$a_1(m,k) = \frac{a(m,k)}{b(m,k)}.$$

The following recurrence formulas are valid:

$$\begin{aligned} a_1(m,k) = a_k + \sum_{r=0}^{m-1} & \frac{1}{b(r+1,k)(r+k/2n+1)(r+k/2n+2l+2)} \\ & \times \left[\sum_{p=-(n-1)}^{n-k-1} B_p \, b(r, p+k+n) \, a_1(r, p+k+n) \right. \\ & \left. + \sum_{p=n-k}^{n-1} B_p \, b(r+1, p+k-n) \, a_1(r+1, p+k-n) \right], \end{aligned} \tag{6}$$

$$m = 1, 2, \dots; \qquad 0 \le k < 2n.$$

In this case the coefficients a_k $(0 \leqslant k < 2n)$ are determined by the relations

$$a_0 = 1, \qquad a_\nu = \frac{1}{(\nu/2n)(\nu/2n + 2l + 1)} \sum_{p=n-\nu}^{n-1} B_p a_{\nu+p-n} \qquad (1 \leqslant \nu < 2n)$$

and, moreover, the limit exists

$$\begin{aligned} \lim_{m\to\infty} a_1(m,k) = T_k = a_k + \sum_{r=0}^{\infty} & \frac{1}{b(r+1,k)(r+k/2n+1)(r+k/2n+2l+2)} \\ & \times \left[\sum_{p=-(n-1)}^{n-k-1} B_p b(r, p+k+n) a_1(r, p+k+n) \right. \\ & \left. + \sum_{p=n-k}^{n-1} B_p b(r+1, p+k-n) a_1(r+1, p+k-n) \right]. \end{aligned} \tag{7}$$

It is clear that T_k is a function of ε and l.

Theorem 1. *If in equation (1) the function $V(r)$ has the form (4), then*

$$C(\varepsilon, l) = \sum_{k=0}^{2n-1} (2\varepsilon)^{-l-1-A/\varepsilon-k/2n} \frac{\Gamma(1+k/2n)\Gamma(k/2n+2l+2)}{\Gamma(l+1-A/\varepsilon+k/2n)} T_k(\varepsilon, l), \quad (8)$$

where $\Gamma(\alpha)$ is the Euler gamma function.

Analysis of the singularities of the functions $T_k(\varepsilon, l)$ leads to the assertion: the function $C(\varepsilon, l)$, by means of equality (8), can be analytically continued beyond the domain $\operatorname{Re}\varepsilon > 0$, $\operatorname{Re} l > -1/2$. The continued function $C(\varepsilon, l)$ is analytic in the ε-plane, except for the cut line $\varepsilon = iy$ $(y \geqslant 0)$, and is meromorphic in the l-plane, all its poles being simple and located at the points

$$l = \frac{1}{2} - \frac{q}{2} - \frac{p}{4n}, \qquad q = 0, 1, 2, \ldots; \quad 0 \leqslant p \leqslant 2n-1.$$

2. Let in equation (1) $A > 0$. Then, as is known, the discrete spectrum of this equation lies on the negative half-axis and 0 is its only accumulation point. Assuming $l > 1/2$, let us renumber the points of the discrete spectrum in increasing order: $E_1(l), E_2(l), \ldots$. In this case it is obvious that $C(\varepsilon_k, l) = 0$, $\varepsilon_k = \sqrt{-2E_k}$. The question arises whether the functions $E_1(l), E_2(l), \ldots$ can be analytically continued into the complex domain.

Using the results of the preceding subsection, one can obtain the assertion:

Theorem 2. *Let in equation (1)*

$$V(r) = \int_{3/2}^{2} \frac{d\sigma(t)}{r^t},$$

where $\sigma(t)$ is a function of bounded variation. Suppose, moreover, that the majorant

$$\overline{V}(r) = \int_{3/2}^{2} \frac{|d\sigma(t)|}{r^t}$$

is such that

$$\int_0^\delta \overline{V}(r)\, r\, dr < \infty, \quad \delta > 0; \qquad \lim_{r\to\infty} \overline{V}(r)\, r^{3/2} \ln r = 0.$$

Then there exists an M_k such that the k-th eigenvalue $E_k(l)$ of the boundary-value problem (1) for $A > 0$ can be analytically continued to the domain $\operatorname{Re} l \geq$

$-1/2$, $|l+1/2| \geq \delta$, $|l| \geq M_k$. Moreover,

$$E_k(l) = -\frac{1}{2}\frac{A^2}{[l+k-\psi_k(l)]^2}, \qquad |\psi_k(l)| \to 0 \quad \text{as } |l| \to \infty.$$

Corollary. The function $E_k(l)$ is uniquely determined by its values at the integer points $E_k(0), E_k(1), \ldots$

For sufficiently small ε ($\operatorname{Re}\varepsilon \geq 0$) the functions

$$\varepsilon_k(l) = \frac{A}{l+k-\psi_k(l)}$$

have inverses

$$l_k = -k + \frac{A}{\varepsilon} + s_k(\varepsilon), \tag{9}$$

where $|s_k(\varepsilon)| \to 0$ as $\varepsilon \to 0$. Formula (9) gives the law of motion of the Regge poles ([1]) as $\varepsilon \to 0$ ($\operatorname{Re}\varepsilon \geq 0$). If $\operatorname{Re}\varepsilon \geq 0$, then formula (9) remains valid also for $A < 0$.

3. Let us consider potentials of a more general form than a superposition of power-law ones.

Suppose now that $V(r)$ is bounded on any segment $[a, b]$ ($0 < a < b < \infty$) and, moreover:

$$\text{I.} \int_0^\delta r|V(r)|\,dr < \infty. \quad \text{II.} \int_N^\infty |V(r)|\,dr < \infty.$$
$$\text{III.} \lim_{r\to\infty} r|V(r)| = 0. \quad \text{IV.} \lim_{r\to\infty} r^2|V(r)| = 0.$$

Assuming $l > -1/2$, renumber the discrete spectrum of the corresponding equation (1): $E_1(l), E_2(l), \ldots$. Using formula (8) and the comparison theorems, one can estimate the density of the discrete spectrum of equation (1). Here the methods of the theory of functions ([2]) provide us with essential help.

Theorem 3. The following equalities hold:

$$\lim_{r\to\infty}\frac{1}{\ln r}\sum_{\varepsilon_k(l)>1/r}\varepsilon_k(l) = A, \qquad \lim_{r\to\infty}\frac{n(r)}{r} = A,$$

where $n(r)$ is the number of $\varepsilon_k(l)$ $[\varepsilon_k(l) = \sqrt{-2E_k(l)}]$ such that $\varepsilon_k(l) > 1/r$.

It follows from Theorem 3 that, for $l > -1/2$,

$$C(\varepsilon, l) = \prod_{k=1}^{\infty}\left(1 - \frac{\varepsilon_k(l)}{\varepsilon}\right)e^{\varepsilon_k(l)/\varepsilon}e^{g(\varepsilon,l)}, \tag{10}$$

where $g(\varepsilon, l)$ is analytic in the domain $\operatorname{Re}\varepsilon > 0$.

An estimate of the behavior of $g(\varepsilon, l)$ is given by

Theorem 4. For any $\alpha > 1$ and $\delta > 0$ the equality

$$\lim_{|\varepsilon|\to 0} |g(\varepsilon, l)|\,|\varepsilon|^{\alpha} = 0, \qquad -\frac{\pi}{2} + \delta \le \arg\varepsilon < \frac{\pi}{2} - \delta$$

holds.

4. The Schrödinger equation is formulated without taking relativistic effects into account. However, at large values of the energy these effects play an essential role. It is therefore natural to pose the inverse problem for the Schrödinger equation, assuming that certain spectral data are known only on a bounded portion of the spectrum. Let us consider one of the possible formulations of this type.

Let the potential $V(r)$ satisfy the conditions of §3, let $A > 0$, and let $l > -1/2$ be fixed. Denote by E_k the discrete spectrum of equation (1), and by $y_k(r)$ the corresponding normalized eigenfunctions ($\|y_k\| = 1$). Then there exist the limits

$$m_k = \lim_{r\to 0} \frac{y_k(r)}{r^l}, \qquad M_k = \lim_{r\to\infty} y_k(r)\, r^{-A/\varepsilon_k + 1} e^{\varepsilon_k r}.$$

One may assume that $m_k > 0$.

Let the numbers E_k, M_k, m_k $(1 \le k < \infty)$ be known. It is required to find $V(r)$. First of all we find (see (10))

$$g(\varepsilon_k, l) = -\ln\left| M_k m_k \prod_{\substack{p=1 \\ p\ne k}}^{\infty} \left(1 - \frac{\varepsilon_p}{\varepsilon_k}\right) e^{\varepsilon_p/\varepsilon_k} \right| \qquad (\varepsilon_k = \sqrt{-2E_k}).$$

Using Theorems 3, 4 and the generalized Carlson theorem ([2]), we obtain that the function $g(\varepsilon, l)$ is uniquely determined by its values at the points ε_k. Hence $C(\varepsilon, l)$ is also uniquely determined by these data. The information now available is sufficient to find the spectral function $\rho(E)$ and the limiting phase $\delta(E)$.

Thus the problem posed is reduced to the classical inverse problem ([3]). From this fact there follows a procedure for reconstructing $V(r)$ and the following theorem:

Theorem 5. *By the totality of the numbers E_k, M_k, and m_k $(1 \le k < \infty)$, the potential $V(r)$ is determined uniquely.*

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.