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Abstract

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SOLVABLE ALGEBRAIC GROUPS

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Connected algebraic linear groups over an algebraically closed field and over a field of characteristic zero have been studied almost completely by A. Borel ([1]) and C. Chevalley ([2–4]). At the same time, disconnected algebraic groups, with the exception of some general theorems and Abelian groups, have not been studied at all. The condition of connectedness makes it possible to use the powerful methods of algebraic geometry, whereas for disconnected groups topological-geometric methods prove not so effective. In this case one has to carry out a synthetic investigation, combining the methods of algebraic and linear groups. The study of nilpotent algebraic linear groups over a perfect field in ([8]) may serve as confirmation.

In the present article we study the structure of solvable algebraic linear groups over a field of characteristic zero. Establishing the structure of algebraic solvable groups makes it possible to study arbitrary (not necessarily algebraic) linear solvable groups as well. It is shown that an analogous description of the structure of solvable linear groups for fields of positive characteristic is impossible.

With regard to the terminology of the article we shall refer to ([1]) (see also ([8])).

In what follows, all groups under consideration are subgroups of the full linear group $L_n(P)$ of degree n over a field P of characteristic zero.

Definition. A matrix $g \in L_n(P)$ is called **semisimple** (respectively **unipotent**) if all its elementary divisors are simple (the eigenvalues are equal to one). A group $D \subset L_n(P)$, all of whose matrices are semisimple, is called a d**-group** (a generalized torus). By an **algebraic group** is meant a group closed in the Zariski topology over P.

It is known ([7]), Theorem 1), that all matrices of a completely reducible solvable linear group are semisimple; therefore the following is true.

Proposition 1. *Let* Γ *be a solvable group; then the set of all unipotent matrices in* Γ *forms an invariant subgroup* Γ_u.

Remark 1. If the characteristic of the field P is positive, then it is easy to give an example of a solvable group for which Proposition 1 does not hold. This circumstance does not allow one to establish an adequate structure of solvable groups over fields of positive characteristic.

Lemma 1. *Let U be a unipotent algebraic group; then U is a complete torsion-free nilpotent group.*

Proof. It is obvious that U contains no elements of finite order distinct from the identity. Nilpotency follows from the reducibility of U to special triangular form ([5,6]). Let $R(U)$ be the Lie algebra of the group U; then any element $u \in U$ is contained in a one-parameter subgroup $A_\alpha = \{\exp \alpha X\}$, where $\alpha \in P$, $X \in R(U)$ ([2]), Ch. 2, § 13); consequently, there exists $\alpha_0 \in P$ such that $u = \exp \alpha_0 X$. But then $\exp \frac{\alpha_0}{n} X \in U$ for every integer n, which proves the completeness of the group U.

Definition. Let G and A be abstract groups, with A an abelian group written additively. We shall say that a mean with values in A is defined on G if to each function $f(g) \in A$ ($g \in G$) there is assigned an element $\int f(g)\,dg$ having the properties:

1. $\displaystyle\int [f(g) + \varphi(g)]\,dg = \int f(g)\,dg + \int \varphi(g)\,dg.$
2. $\displaystyle\int L[f(g)]\,dg = L\left[\int f(g)\,dg\right]$, where L is an automorphism of A.
3. $\displaystyle\int f(h_1 g h_2)\,dg = \int f(g)\,dg$ for any $h_1, h_2 \in G$.
4. $\displaystyle\int f(g)\,dg = a$, if $f(g) = a$.

For what follows we need the following well-known lemma, whose proof may be found, for example, in ([5]), § 4.

Lemma 2. Let the group G have a solvable normal divisor R with solvable series

$$R \supset R_1 \supset R_2 \supset \cdots \supset R_k \supset (e),$$

such that a mean is defined on the group G/R with values in each factor R_i/R_{i+1}; then G contains a subgroup H with the properties: $H \cap R = (e)$, $HR = G$, and all subgroups H having these properties are conjugate in G by means of elements of R.

It is obvious that if R is a complete torsion-free abelian group and the factor group G/R is finite, then the hypotheses of Lemma 2 are fulfilled. On the other hand, the factors of the upper central series of a complete torsion-free nilpotent group are complete torsion-free abelian groups; hence, from Lemmas 1 and 2 it follows (see ([5]))

Theorem 1. Let G be an algebraic group whose unipotent matrices form an invariant subgroup G_u of finite index; then $G = DG_u$, where D is any maximal finite subgroup of G; all maximal finite subgroups are conjugate in G by means of matrices from G_u.

The main result of the present paper is

Theorem A. A solvable algebraic group Γ can be represented in the form of a semidirect product

$$\Gamma = D\Gamma_u,$$

where D is any maximal d-subgroup (maximal generalized torus) of Γ; all maximal d-subgroups of Γ are conjugate by means of elements of Γ_u.

For the case of a connected group Γ, an analogous theorem was obtained in ([9]). This result will be used essentially in what follows.

We shall preface the proof of Theorem A with several propositions, some of which are of independent interest. The following lemmas are proved quite simply.

Lemma 3. Let Γ be an algebraic solvable group, Γ_0 its connected component; then $\Gamma_u = \Gamma_{0u}$.

Lemma 4. Let $\Gamma = D\Gamma_u$, where D is some d-subgroup of Γ; then D is a maximal d-subgroup of Γ.

Lemma 5. Let

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12}, \dots, \Gamma_{1k} & & \\ 0 & \Gamma_{22}, \dots, \Gamma_{2k} & & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0, \dots, \Gamma_{kk} & & \end{bmatrix}, \tag{1}$$

where Γ_{ii} $(i = 1, 2, \dots, k)$ are irreducible groups; then for any matrix $g \in \Gamma$ there exists a semisimple matrix $g_s \in \Gamma$ such that the irreducible parts of g and g_s coincide.

Theorem 2. Let Γ be a solvable algebraic group whose connected component Γ_0 is nilpotent. Then

$$\Gamma = D\Gamma_u,$$

where D is some maximal d-subgroup of Γ. All maximal d-subgroups of the group Γ are conjugate by means of elements of Γ_u.

Proof. Since Γ_0 is nilpotent, $\Gamma_0 = \Gamma_s^0\Gamma_u$, where Γ_s^0 is the connected algebraic subgroup of all semisimple matrices of the group Γ_0 ([1], [9]). It follows that Γ_s^0 is a normal divisor in Γ. Let f be a rational representation of the group Γ whose kernel is Γ_s^0. The image of a semisimple (unipotent) matrix under a rational representation is a semisimple (respectively unipotent) matrix (([1]), Theorem 9.3). Since P is a field of characteristic zero, the image of a unipotent algebraic group under a rational representation is a unipotent algebraic group ([3], Ch. 5). Let $f(\Gamma) = \Gamma'$; then $f(\Gamma_u) = \Gamma'_u$, and it is easy to see that the index of Γ'_u in Γ' is finite, i.e., the group Γ' is algebraic. It is clear that Γ has maximal d-subgroups, and every semisimple matrix is contained in some maximal d-subgroup. Let D_1 and D_2 be maximal d-subgroups of Γ; then it is clear that $f(D_1) = D'_1$ and $f(D_2) = D'_2$ are maximal finite subgroups in Γ'. By Theorem 1, there then exists a matrix $t_1 \in \Gamma'_u$ such that $t_1^{-1}D'_1t_1 = D'_2$. Hence $t^{-1}D_1t = D_2$, where $t = f^{-1}(t_1) \in \Gamma_u$. We may assume that the group Γ has been brought to the form (1); then all maximal d-subgroups have identical irreducible parts, and from Lemma 5 it follows that all maximal d-subgroups are isomorphic to the d-group

$$H = \begin{pmatrix} \Gamma_{11} & 0 & \dots & 0 & \\ 0 & \Gamma_{22} & \dots & 0 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \Gamma_{kk} & \end{pmatrix},$$

which, apparently, proves Theorem 2.

Theorem 3. *If*

$$\Gamma = D\Gamma_u, \tag{2}$$

where D is some d-subgroup of Γ, then all d-subgroups possessing property (2) are conjugate by means of a matrix $t \in \Gamma_u$.

Proof. Let D_1 and D_2 satisfy (2). By Lemma 4, D_1 and D_2 are algebraic maximal d-subgroups of Γ. Denote by Q_1 and Q_2 the connected components of D_1 and D_2, respectively. It is clear that Q_1 and Q_2 are maximal tori in the connected component Γ_0 of the group Γ (Lemma 3). Let N_1 and N_2 be the normalizers of Q_1 and Q_2 in the group Γ. Then N_1 and N_2 are algebraic groups whose connected components N_1^0 and N_2^0 are nilpotent. Indeed, $N_1^0 = Q_1 N_{1u}$, $N_2^0 = Q_2 N_{2u}$, where the products are direct; consequently, Q_1 and Q_2 lie in the centers of N_1 and N_2, respectively. By Theorem 2 from ([9]), there exists $t_1 \in \Gamma_u$ such that $t_1^{-1} Q_1 t_1 = Q_2$. It follows that $t_1^{-1} N_1 t_1 = N_2$. At the same time, by Theorem 2 of the present work, all maximal d-subgroups in N_1 and N_2 are conjugate by means of matrices from Γ_u. Since D_1 and D_2 are maximal d-subgroups, respectively in N_1 and N_2, it follows that there exists a matrix $t \in \Gamma_u$ such that $t^{-1} D_1 t = D_2$.

Remark 2. Theorems 1 and 3 of the present work are not true for arbitrary solvable linear groups. This is easy to verify if, as the group Γ, one takes the group of matrices of the form $\begin{pmatrix} \pm 1 & k \\ 0 & \pm 1 \end{pmatrix}$, where $k = 0, \pm 1, \pm 2, \dots$.

Theorem 4. *In a solvable algebraic group Γ there exists a maximal d-subgroup D such that $\Gamma = D \cdot \Gamma_u$.*

Proof. Let Q be some maximal torus of the connected component Γ_0, and let N be the normalizer of Q in the group Γ. We may assume that the group Γ has the form (1). We shall show that N contains semisimple matrices with all irreducible parts occurring in Γ.

Let $g \in \Gamma$; then $gQg^{-1} = Q_g$ is a maximal torus in Γ_0. From ([9]) it then follows that there exists $t \in \Gamma_u$ such that $tQ_{gt}^{-1} = Q$, and therefore $tgQg^{-1}t^{-1} = Q$, whence $t \cdot g \in N$. Since g is an arbitrary matrix, N contains matrices, therefore, by Lemma 5, also semisimple matrices with all possible irreducible components. As in Theorem 3, N is an algebraic group with nilpotent connected component; therefore, by Theorem 2, all maximal d-subgroups of the group N are conjugate in N and are isomorphic to the group H. Hence it follows that $\Gamma = D\Gamma_u$, where D is any maximal d-subgroup of N.

From Theorems 3 and 4 it follows that

Theorem 5. *Every d-subgroup M of the group Γ is contained in some d-subgroup D_M having the property $\Gamma = D_M\Gamma_u$.*

Corollary 1. *If G is a connected solvable algebraic group, then every Abelian d-subgroup H is contained in some maximal torus Q_H of the group G.*

It can be shown that Corollary 1 is valid for an arbitrary perfect field P.

The proof of Theorem A now follows in an obvious way from Theorems 3, 4, and 5.

In ([10]) the structure of arbitrary locally nilpotent linear groups over a perfect field was established. Since the closure of a solvable group in the Zariski topology is solvable, Theorem A implies the analogous result for solvable linear groups over a field of characteristic zero.

Theorem B. *Every solvable linear group G can be embedded in a solvable group $\overline{G}$ that is a semidirect product $\overline{G} = D\overline{G}_u$, where D is a maximal d-subgroup of $\overline{G}$.*

All maximal d-subgroups of the group $\overline{G}$ are conjugate in $\overline{G}$.

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CITED LITERATURE

1. A. Borel, *Ann. of Math.*, **64**, No. 1, 20 (1956).
2. C. Chevalley, *Théorie des groupes de Lie*, **2**, IL, 1958.
3. C. Chevalley, *Théorie des groupes de Lie*, **3**, IL, 1958.
4. C. Chevalley, *Séminaire École Normale Supérieure*, Paris, 1956–1958.
5. A. I. Mal' tsev, *Matem. sborn.*, **28** (70), No. 3, 567 (1951).

6. E. Kolchin, *Ann. of Math.*, **49**, 774 (1948).

7. D. A. Suprunenko, *Dokl. AN BSSR*, **6**, No. 8 (1961).

8. V. P. Platonov, *DAN*, **146**, No. 5 (1962).

9. V. P. Platonov, *Dokl. AN BSSR*, **7**, No. 12 (1962).

10. D. A. Suprunenko, R. I. Tyshkevich, *Izv. AN SSSR, ser. matem.*, **24**, 787 (1960).

Note: Figure translations are in progress. See original paper for figures.