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Abstract

Full Text

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(L_p, L_q)-MULTIPLIERS OF FOURIER INTEGRALS

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1. Let the function $f(x)$ be defined and summable to the power p in E_n $(f(x) \in L_p,\ 1 < p < \infty)$, and let $\tilde{f}(\lambda)$ be its Fourier transform

$$\tilde{f}(\lambda) = \frac{1}{(2\pi)^{n/2}} \int_{E_n} f(x) e^{-i\lambda x}\, dx, \qquad f \in S^*. \tag{1}$$

Hörmander ([1]) showed that every bounded operator A from L_p into L_q $(A \in L_p^q)$, commuting with translations, is represented, for $p \le q < \infty$, by a convolution $A\varphi = T * \varphi;\ \varphi \in S,\ T \in S'$, which in Fourier images takes the form $\widetilde{A\varphi} = \widetilde{T}\widetilde{\varphi}$.

Under these conditions the distribution $\widetilde{T}$ is called a multiplier of type (p, q), and the set $\{\widetilde{T};\ T \in L_p^q\}$ is denoted by M_p^q. The distribution $\widetilde{T} \in M_p^q$ is a locally summable function ([1]).

In the present paper we are interested in conditions on a function $\Phi(\lambda)$ which ensure that it belongs to M_p^q. Conditions of this kind, given in the case $p = q$ by S. G. Mikhlin ([2]), have proved useful in many questions. We follow the method set forth in ([3]) and originating with Marcinkiewicz ([4])**. Our results generalize the results of the works mentioned (even for $p = q$).

2. Main theorem. *Let the function $\Phi(\lambda)$ be continuous together with the derivative $\partial^n \Phi / \partial\lambda_1 \cdots \partial\lambda_n$ and all preceding derivatives outside the coordinate planes (i.e. for $|\lambda_j| > 0,\ j = 1, \dots, n$). Then $\Phi(\lambda) \in M_p^q$, if for the derivatives just mentioned*

$$\left| \lambda_1^{k_1+\beta} \cdots \lambda_n^{k_n+\beta} \frac{\partial^k \Phi}{\partial\lambda_1^{k_1} \cdots \partial\lambda_n^{k_n}} \right| \le M, \tag{2}$$

where $\beta = 1/p - 1/q,\ k_j$ takes the value 0 or 1, $k = \sum_{j=1}^n k_j = 0, 1, \dots, n,$ and M is a constant.

This theorem remains valid if one considers vector-valued functions $f(x)$ from $L_p(H)$ (i.e. with values in an arbitrary Hilbert space H and with finite norm $\|f\|_{L_p(H)} = \int_{E_n} \|f(x)\|_H\, dx$). In this case $\Phi(\lambda)$ is an operator-valued function (i.e. for each $\lambda \in E_n$, $\Phi(\lambda)$ is a bounded operator in H), whose derivatives are understood in the strong sense; condition (2) is written in terms of the operator norm, and it is asserted that

$$\left\|\widehat{\Phi(\lambda)\tilde{f}(\lambda)}\right\|_{L_q(H)} \leq c\|f\|_{L_p(H)} *** .$$

First of all we shall reproduce the main steps of the proof of the theorem in the scalar case for $n = 1,\ p < q$.

3. Let $f(x) \in L_p(l_2)$, i.e. $f(x) = \{f_k(x)\}_{k=1}^{\infty}$ and

$$\|f\|_{L_p(l_2)} = \left\{\int_{-\infty}^{\infty}\left[\sum_{1}^{\infty}|f_k(x)|^2\right]^{p/2} dx\right\}^{1/p} < \infty.$$

* $S\,(S')$ is the space of Schwartz test (generalized) functions.
** The work ([4]) concerns series; S. G. Mikhlin derived the mentioned criterion ([2]) by passing from the conditions obtained in ([4]).
*** $\hat{g}$ is the inverse Fourier transform of the function g.

Lemma 1. For every $p,\ 1 < p < \infty$, the convolution transform

$$g(x) = \mathcal{H}_0 f \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{f(y)\,dy}{|x-y|^{\alpha}} \tag{3}$$

defines a bounded transformation from $L_p(l_2)$ into $L_q(l_2)$, where $1/p - 1/q = 1-\alpha,\ p < q < \infty$.

The proof of Lemma 1 reduces to applying Theorem 2 from [3]. In what follows we shall use the theorem on the Fourier transform of a convolution:

$$\widetilde{h * f} = \tilde{h}\cdot\tilde{f},$$

regarding h as a distribution from S' and $h * f$ as $\{h * f_k\}_1^{\infty}$. The Fourier transform of the function $f(x) \in L_p(l_2)$ may here be understood in the sense

$$\tilde{f}(\lambda) = \{\tilde{f}_k(\lambda)\}_1^{\infty}.$$

Lemma 2. Let $f \in L_p(l_2)$, $1 < p < \infty$, and let $\Phi(\lambda)$ be defined by the formula

$$\Phi(\lambda) = \int_{-\infty}^{\infty}\frac{d\rho(t)}{|t-\lambda|^{\beta}}, \qquad \beta = \frac{1}{p} - \frac{1}{q}, \qquad p < q < \infty,$$

where $\rho(t)$ is a function of bounded variation. Then the transformation $\mathcal{K}$, defined in Fourier images by the formula

$$\widetilde{\mathcal{K}f} = \Phi(\lambda)\tilde{f}(\lambda),$$

is a bounded mapping from $L_p(l_2)$ into $L_q(l_2)$.

Proof. Let $f \in S$ (i.e., the functions $f_k(x)$, $k = 1, 2, ...$, are infinitely differentiable and decrease at infinity faster than any power of $|x|$). We have

$$\begin{aligned}\mathcal{K}f &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(\lambda)\tilde{f}(\lambda)e^{i\lambda x}\,d\lambda \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(\lambda)e^{i\lambda x}\left(\int_{-\infty}^{\infty}\frac{d\rho(t)}{|\lambda - t|^{\beta}}\right)d\lambda \\ &= \int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{|\lambda - t|^{\beta}}\tilde{f}(\lambda)e^{i\lambda x}\,d\lambda\right)d\rho(t) \\ &= \int_{-\infty}^{\infty}\mathcal{H}_t f\,d\rho(t).\end{aligned} \tag{4}$$

Here by $\mathcal{H}_t$ we have denoted the transform defined by the formula

$$\widetilde{\mathcal{H}_t f} = \frac{1}{|\lambda - t|^{\beta}}\tilde{f}(\lambda).$$

It maps $L_p(l_2)$ continuously into $L_q(l_2)$. Indeed, from formula (3) we have

$$\widetilde{\mathcal{H}_0 f} = |x\widehat{|^{-\alpha} * f} = D(\alpha)|\lambda|^{-1+\alpha}\tilde{f}, \qquad 1 - \alpha = \frac{1}{p} - \frac{1}{q} \ \ * .$$

The transform $\mathcal{H}_t$ differs from $\mathcal{H}_0$ (up to the multiplicative constant $D(\alpha)$) only by a shift of the first factor by t. Since $\mathcal{H}_0$ is bounded, $\mathcal{H}_t$ is also bounded (independently of t). From relation (4) we easily obtain

$$\|\mathcal{K}f\|_{L_q(l_2)} \le \int_{-\infty}^{\infty}\|\mathcal{H}_t f\|_{L_q(l_2)}\,|d\rho(t)| \le c\|f\|_{L_p(l_2)},$$

and Lemma (2) follows from the density of S in $L_r(l_2)$, $1 < r < \infty$.

Lemma 3. Let $f \in L_p(l_2)$, $1 < p < \infty$, and let $\Phi_m(\lambda)$ be a sequence of functions representable in the form

$$\Phi_m(\lambda) = \int_{-\infty}^{\infty}\frac{d\rho_m(t)}{|\lambda - t|^{\beta}}, \qquad \beta = \frac{1}{p} - \frac{1}{q}, \qquad p < q < \infty, \tag{5}$$

where $\rho_m(t)$ are functions of bounded variation for which

$$\underset{-\infty<t<\infty}{\mathrm{var}} \rho_m(t) \leq M.$$

Then the mapping

$$\widetilde{(\mathcal{K} f)}_m = \Phi_m(\lambda) \tilde{f}_m(\lambda)$$

is a bounded mapping from $L_p(l_2)$ into $L_q(l_2)$.

* The factor $D(\alpha) = -\sqrt{2\pi}/2 \cos \dfrac{\pi\alpha}{2}\, \Gamma(\alpha)$ does not vanish for the α under consideration.

Proof. First the boundedness of the "truncated" operator $\mathcal{K}_N$ is proved:

$$(\widetilde{\mathcal{K}_N \mathbf{f}})_m = \begin{cases} \Phi_m(\lambda)\, \tilde{f}_m(\lambda), & m \leqslant N, \\ 0, & m > N. \end{cases}$$

By the example of the preceding lemma, $\mathcal{K}_N$ is represented in the form of a superposition

$$\mathcal{K}_N \mathbf{f} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathcal{H}(t_1, \dots, t_N) \mathbf{f}\, dp_1(t_1) \cdots dp_N(t_N) \tag{6}$$

of bounded operators $\mathcal{H}(t_1, \dots, t_N)$, defined by the equality

$$\mathcal{H}((t_1, \dots, t_N) \mathbf{f} = \begin{cases} \dfrac{1}{|\lambda - t_m|^\beta}\, \tilde{f}_m, & m \leqslant N, \\ 0, & m > N. \end{cases}$$

Since

$$\mathcal{H}(t_1, \dots, t_N) = \mathcal{M}(t_1, \dots, t_N)\, \mathcal{H}(0, \dots, 0)\, \mathcal{M}(-t_1, \dots, -t_N),$$

where $\mathcal{M}(t_1, \dots, t_N)$ is an isometry of $L_r(l_2)$, $1 < r < \infty$, given by the formula

$$f_m(x) \to e^{it_m x} f_m(x), \qquad t_m = 0 \text{ for } m > N,$$

and $\mathcal{H}(0, \dots, 0)$ coincides with the restriction of $\mathcal{H}_0$ to the subspace of dimension N, the norm of $\mathcal{H}(t_1, \dots, t_N)$ is bounded (uniformly with respect to N and independently of t_j).

If now

$$\underset{-\infty<t<\infty}{\mathrm{var}} \rho_m(t) = 1, \qquad m = 1, 2, \dots,$$

then from (6) the boundedness of $\mathcal{K}_N$, independently of N, follows, and as $N \to \infty$ we obtain the assertion of the lemma. In the general case one must normalize the functions ρ_m (to unit total variation) by multiplying by certain constants a_m; since

$$c_m \geqslant \delta > 0 \quad (\text{for } \operatorname{var} \rho_m(t) \leqslant M),$$

the proof is not affected.

4. The proof of our main theorem is based on Lemma 3 and on the following proposition (see ([5],[3])).

Theorem 1 (on decompositions). Let $f(x) \in L_p$, $1 < p < \infty$, and let the Fourier transform of the function $f_m(x)$ be concentrated in the interval

$$2^m < |\lambda| \leqslant 2^{m+1}$$

and coincide there with $\tilde{f}(x)$. Then there exist constants c_1 and c_2, independent of f, such that

$$c_1 \|f\|_{L_p} \leqslant \left\{ \int_{-\infty}^{\infty} \left(\sum_{-\infty}^{\infty} |f_m(x)|^2 \right)^{p/2} \right\}^{1/p} \leqslant c_2 \|f\|_{L_p}.$$

Finally, suppose that we are given a function $\Phi(\lambda)$, differentiable away from the origin and such that

$$|\Phi(\lambda)|\,|\lambda|^{\beta} \leqslant M, \qquad |\Phi'(\lambda)|\,|\lambda|^{1+\beta} \leqslant M. \tag{7}$$

We shall prove that $\Phi(\lambda) \in M_p^q$. The mapping $\mathcal{L}$ defined by Φ will be represented in the form

$$\mathcal{L} = A_3 A_2 A_1,$$

where the linear operators A_1, A_2, A_3 act according to the scheme

$$L_p \xrightarrow{A_1} L_p(l_2) \xrightarrow{A_2} L_q(l_2) \xrightarrow{A_3} L_q^*$$

and are defined as follows:

$$A_1 \mathbf{f} = \mathbf{f} = \{f_m\}_{-\infty}^{\infty} \quad (f_m \text{ is connected with } f \text{ as in Theorem 1});$$

$$\mathbf{g} = A_2 \mathbf{f} = \{\Phi_m \tilde{f}_m\}_{-\infty}^{\infty}, \qquad \Phi_m(\lambda) = \begin{cases} \Phi(\lambda), & \text{for } 2^m < |\lambda| \leqslant 2^{m+1}, \\ 0, & \text{for } \lambda \notin (2^m, 2^{m+1}]; \end{cases}$$

$$\widetilde{A_3 \mathbf{g}} = \Phi(\lambda) \tilde{f}(\lambda) \quad (= \Phi_m(\lambda) \tilde{f}_m(\lambda),\ |\lambda| \in (2^m, 2^{m+1}];\ m = 0, \pm 1, \ldots).$$

The boundedness of the operators A_1 and A_3 follows from Theorem 1, and it remains for us to prove the boundedness of the operator A_2 under conditions (7). According to Lemma 3, for this it is enough to verify that the functions $\Phi_m(\lambda)$ are representable in the form (5). Thus the question is reduced to the solvability of the integral equation

$$\int_{-\infty}^{\infty} \frac{dp_m(t)}{|\lambda - t|^{\beta}} = \Phi_m(\lambda) = \begin{cases} \Phi(\lambda), & \text{for } 2^m < |\lambda| \leqslant 2^{m+1}, \\ 0, & \text{for } |\lambda| \notin (2^m, 2^{m+1}] \end{cases} \tag{8}$$

The solution of this equation is given by the formula

$$\rho'_m(\lambda) = b\left\{\int_{2^m}^{2^{m+1}} \frac{\Phi'(t)}{|t-\lambda|^\alpha}\operatorname{sign}(t-\lambda)\,dt + \Phi(2^m)\frac{\operatorname{sign}(2^m-\lambda)}{|2^m-\lambda|^\alpha} - \Phi(2^{m+1})\frac{\operatorname{sign}(2^{m+1}-\lambda)}{|2^{m+1}-\lambda|^\alpha}\right\}.$$

A calculation shows that the function $\rho'_m(\lambda)$ is summable and, for a certain constant c independent of Φ, the inequality

$$\operatorname*{var}_{-\infty<\lambda<\infty} \rho_m(\lambda) = \int_{-\infty}^{\infty} |\rho'_m(t)|\,dt \le cM$$

holds.

The proof of the theorem in the case under consideration is complete.

5. In the n-dimensional case, as the "elementary" mapping $\mathcal{H}_0$ (see Lemma 1) from $L_p(l_2)$ into $L_q(l_2)$, one should consider the mapping

$$\mathcal{H}_0 f = \frac{1}{(2\pi)^{n/2}}\int_{E_n} \frac{\operatorname{sign}(x_1-y_1)\cdots\operatorname{sign}(x_n-y_n)}{|x_1-y_1|^\alpha\cdots|x_n-y_n|^\alpha} f(y)\,dy, \tag{9}$$

where $1/p - 1/q = 1-\alpha, \quad p \le q < \infty$. The introduction here of the factor

$$\prod_1^n \operatorname{sign}(x_i - y_i)$$

is caused by the desire to include also the singular case $\alpha = 1$

$$\left(\text{in this case the integral in (9) is understood in the sense of } \lim_{\sum \delta_i^2\to 0}\int_{|x_i-y_i|>\delta;\ i=1,\dots,n} \cdots dy\right).$$

Instead of the sequence (5) one has to consider a sequence with n "inputs"

$$\Phi_{m_1,\dots,m_n}(\lambda) = \sum \underbrace{\int\cdots\int}_{s} \frac{d\rho_{m_{j_1},\dots,m_{j_s}}(t_{j_1},\dots,t_{j_s})}{|\lambda_{j_1}-t_{j_1}|^\beta\cdots|\lambda_{j_s}-t_{j_s}|^\beta}, \tag{10}$$

where the summation ranges over all subspaces of E_n of dimension s, $1 \le s \le n$ ($(t_{j_1},\dots,t_{j_s})$ are the coordinates of a point of the subspace over which the integration is performed); $\rho_{m_{j_1},\dots,m_{j_s}}$ are finite measures whose total variations are uniformly bounded. The splitting theorem takes the form

$$c_1\|f\|_{L_p} \le \left\{ \int_{E_n} \left(\sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_n=-\infty}^{\infty} |f_{m_1,\dots,m_n}(x)|^2 \right)^{p/2} \right\}^{1/p} \le c_2\|f\|_{L_p}, \quad (11)$$

where the function $f_{m_1,\dots,m_n}$ is defined by the requirement that its Fourier transform be concentrated in the region $2^{m_j} < |\lambda_j| \le 2^{m_j+1}$, $j = 1, \dots, n$, and coincide there with $\hat{f}(\lambda)$. The arguments of item 4 remain valid with equation (5) replaced by equation (10); the solution of the latter is likewise written explicitly and satisfies the required conditions.

6. The transfer of the proof to H-valued functions is carried out following the example of the work [3]. It should be noted that the splitting theorem proved in [3] for $n = 1$ is also valid in the case of arbitrary n in the form (11) (and for H-valued functions).

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Note: Figure translations are in progress. See original paper for figures.