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# MATHEMATICS

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**Abstract**

**Full Text**

## MATHEMATICS

**B. P. KUFAREV, S. V. SOBOLEVA**

### A CONTINUUM AS THE COMPLETE LIMIT SET OF A CONVERGENT SEQUENCE OF ANALYTIC FUNCTIONS

*(Presented by Academician M. A. Lavrent'ev, 28 VI 1963)*

1. Let an arbitrary continuum  $K$ , not containing the point  $w_0$ , be given in the plane  $R_2$  of the complex variable  $w$ . Our main result is

**Theorem 1.** *If the point  $w_0$  belongs to the component of the set  $R_2 \setminus K$  that does not contain  $\infty$ , then there exists a sequence of functions  $\{f_n(z)\}$ , analytic in  $D$ , continuous and single-valued in  $D$ , converging uniformly inside the disk  $D: |z| < 1$ , with  $f_n(0) = w_0$ ,  $f'_n(0) > 0$ , such that the set of all limit points of all possible sequences of the form  $\{f_n(z_n)\}$ ,  $n = 1, 2, \dots$ ,  $\lim_{n \rightarrow \infty} z_n = z_0$ ,  $|z_n| < 1$ , coincides with  $K$ , whatever the point  $z_0$  on the circumference  $|z| = 1$ .*

*If, however,  $w_0$  belongs to the component of the set  $R_2 \setminus K$  containing  $\infty$ , then the same result holds, except that each function  $f_n$  has a simple pole in the disk.*

Using the terminology and results of the theory of prime ends of a simply connected domain (after Carathéodory <sup>(1)</sup>) and the theory of prime ends of a sequence of plane domains converging to a kernel, constructed by G. D. Suvorov in <sup>(2)</sup>, the proof of this theorem can be reduced to the construction of a sequence of simply connected domains  $B(B_{w_0}) \equiv \{B_n\}$ , converging to the kernel  $B_{w_0}$ , with Jordan boundaries ( $n = 1, 2, \dots$ ), such that the body of each prime end of this sequence coincides with  $K^*$ .

Without giving this construction here, we note only that by a simple modification of it one can obtain the result of <sup>(3)</sup>. We also note that it is not known to us whether there exists a bounded simply connected plane domain all of whose prime ends, in the sense of Carathéodory, have one and the same body, whereas the method used in the proof of Theorem 1 makes it easy to construct a sequence of domains converging to a kernel, all of whose prime ends are of the first type and have one and the same body, coinciding with the boundary of the kernel.

2. Our result can be formulated differently (see Corollary 2 below) if the following two definitions are introduced:

**Definition 1.** The **complete limit set** ( $C\{f_n\}$ ) of a sequence of functions  $\{f_n(z)\}$ ,  $n = 1, 2, \dots$ , defined in the disk  $D$ , is the set of all limit points of all

possible sequences of the form  $\{f_n(z_n)\}$ , where  $z_n \in D$ ,  $\lim_{n \rightarrow \infty} |z_n| = 1$ .

If, in addition to the last two conditions,  $z_n \in G$ , where  $G$  is a set lying in the disk whose closure intersects the unit circumference, then we have the partial limit set  $C_G(\{f_n\})$  of the sequence  $\{f_n(z)\}$  (with respect to the set  $G$ ).

Analogously, by adding the condition  $\lim_{n \rightarrow \infty} z_n = z_0$  (respectively  $z_n \in G \subset D$ ,  $\overline{G} \ni z_0$ ,  $\lim_{n \rightarrow \infty} z_n = z_0$ ), one defines the limit set  $C(\{f_n\}, z_0)$  (respectively the partial limit set  $C_G(\{f_n\}, z_0)$ ) of the sequence  $\{f_n\}$  at the point  $z_0$ .

\* The possibility of such a construction was conjecturally suggested by A. I. Prilepko.

Obviously,  $C_G(\{f_n\}) \subseteq C(\{f_n\})$  and  $C_G(\{f_n\}, z_0) \supseteq C(\{f_n\}, z_0)$ .

**Definition 2.** We say that  $G$  is a **set of maximal indeterminacy as a whole** if  $C_G(\{f_n\}) = C(\{f_n\})$ , and a **maximal indeterminacy at the point**  $z_0$ , if  $C_G(\{f_n\}, z_0) = C(\{f_n\}, z_0)$ \*.

We give two immediate corollaries of our theorem.

**Corollary 1.** An arbitrary continuum  $K \ni w_0$  (not coinciding with the plane) is the complete limit set of some sequence of univalent analytic\*\* functions  $\{f_n(z)\}$ ,  $f_n(0) = w_0$ ,  $f'_n(0) > 0$ , converging uniformly inside the disk  $|z| < 1$ .

**Corollary 2.** There exists a sequence of univalent analytic functions  $\{f_n(z)\}$ ,  $f_n(0) = w_0$ ,  $f'_n(0) > 0$ , converging uniformly inside the disk  $|z| < 1$ , such that

$$C(\{f_n\}, z_0) = C(\{f_n\}) = K,$$

whatever boundary point  $z_0$  of the disk may be; consequently, every neighborhood of any boundary point  $z_0$  of the disk is a set of maximal indeterminacy as a whole for the sequence  $\{f_n\}$ .

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\* The corresponding definitions for a single function are given, for example, in paper (4).

\*\* Possibly with one simple pole.

*Note: Figure translations are in progress. See original paper for figures.*

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