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Abstract

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On a New Criterion for the Speciality of πd-Groups with a Given Number of Classes of Unattainable Isoordinal πd-Subgroups

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§ 1. In the present paper we mainly use the notation, concepts, and definitions of our preceding notes ([1–3]). In addition, we shall use the terminology of papers ([4,5]), concerning isoordinal subgroups and classes of isoordinal attainable and unattainable subgroups. The number of classes of unattainable isoordinal πd-subgroups ([6]) we shall agree to denote by r, and the number of distinct prime π-divisors of the order of the group G by t.

In paper ([7]) a theorem had already been obtained on the solvability of a group G for which $r = t - 1$, and in paper ([5])—on the π-solvability of G satisfying the equality $r = t$, and on the π-separability of G with the relation $r = t + 1$. The latter results generalized and strengthened Theorems 2 and 3 ([1]), ([8–10]) and Theorems 4 and 5 ([11]). From them, as corollaries, we obtained theorems on the π-solvability of groups for which $r = 1$, and on the π-separability of groups with $r = 2, 3$ ([7]).

In the present work a simpler method is given for proving theorems on the solvability of πd-groups for which $r = t - 1$, than the method of obtaining the analogous result in papers ([7,8]), and Theorem 1 and Corollaries 1 and 2 ([5]) are also strengthened. With the help of the theorems of § 2 a new criterion for the speciality of πd-groups is obtained, following from Theorems 3 and 4.

§ 2. We give proofs of the theorems mentioned.

Theorem 1. *If a πd-group G satisfies the relation $r = t - 1$, then it is solvable.*

Proof. Since a special group is always solvable, suppose that G is a nonspecial group. Obviously, among all its Sylow subgroups $P_1, P_2, \ldots, P_t$ there is at least one attainable. In the sequel, unattainable Sylow πd-subgroups of distinct orders will everywhere be denoted by the first s indices, and attainable ones by the remaining $s + 1, s + 2, \ldots, t$ indices.

We consider separately each of the possible cases:

$$1) \quad s = 0, \qquad 2) \quad 0 < s < r, \qquad 3) \quad s = r.$$

1) $s = 0$, i.e. $P_1, P_2, \ldots, P_{r+1}$ are attainable in G.

By Lemma 1 ([4]) all of them are invariant in G. In view of the nonspeciality of G, at least one q-subgroup, for example Q_1, is noninvariant in it. Form the collection of πd-subgroups:

$$P_1Q_1,\ P_2Q_1, \dots,\ P_{r+1}Q_1, \tag{*}$$

among which one, for definiteness $P_{r+1}Q_1$, will necessarily be attainable and invariant in G (Lemma 3 ([11]) and Lemma 1 ([4])). Obviously, the πd-subgroups $P_{rP_{r+1}}Q_1, \dots,\ \dots, P_1 \cdots P_{rP_{r+1}}Q_1,\ P_1 \cdots P_{rP_{r+1}}Q_1Q_2, \dots,$ $P_1 \cdots P_{rP_{r+1}}Q_1Q_2 \cdots Q_{\nu-1}$, will also be attainable and, consequently, invariant; these form the normal series

$$G \supset P_1 \cdots P_{rP_{r+1}}Q_1Q_2 \cdots Q_{\nu-1} \supset \cdots \supset P_1 \cdots P_{rP_{r+1}}Q \supset \cdots \supset P_{rP_{r+1}}Q_1 \supset P_{r+1}Q_1 \supset P_{r+1} \supset E,$$

i.e. G is solvable.

2) $0 < s < r$, i.e. $P_1, P_2, \dots, P_s$ are unattainable, while $P_{s+1}, P_{s+2}, \dots, P_{r+1}$ are attainable in G. Then among the $r + 1 - s > 1$ πd-subgroups of the newly formed collection

$$P_{s+1}P_1,\ P_{s+2}P_1, \dots,\ P_{r+1}P \tag{**}$$

only one subgroup is attainable in G. Moreover, it is obvious that $s < 1$. Since the case $s = 0$ has been dealt with, put $s = 1$. Let P_2P_1 be attainable in G. In view of the presence in G of all r classes of unattainable isoordinal πd-subgroups, the products $P_3P_2P_1, \dots, P_{r+1} \cdots P_3P_2P_1,\ P_{r+1} \cdots P_3P_2P_1Q_1,$ $P_{r+1} \cdots P_3P_2P_1Q_1Q_2, \dots,\ P_{r+1} \cdots P_3P_2P_1Q_1Q_2 \cdots Q_{\nu-1}$ form a normal series of the required kind, i.e. G is soluble.

3) $s = r$, i.e. $P_1, P_2, \dots, P_s$ are unattainable, while P_{s+1} is attainable in G (here $r + 1 - s = 1$).

Obviously, P_{s+1} is invariant in G. Since G already has all classes of unattainable isoordinal πd-subgroups, the products $P_{sP_{s+1}}, \dots, P_2 \cdots P_{sP_{s+1}},$ $P_1P_2 \cdots P_{sP_{s+1}},\ P_1P_2 \cdots P_{sP_{s+1}}Q_1, \dots,\ P_1P_2 \cdots P_{sP_{s+1}}Q_1Q_2 \cdots Q_{\nu-1}$ will be attainable and invariant πd-subgroups (by Lemma 1 ([4])), forming a normal series $G \supset P_1P_2 \cdots P_{sP_{s+1}}Q_1Q_2 \cdots Q_{\nu-1} \supset \cdots \supset P_{sP_{s+1}} \supset P_{s+1} \supset E$, i.e. G is soluble.

As follows from the proof, the theorem formulated is also true in the case when all prime divisors of the order of the group belong to the set π.

Theorem 2. *If a πd-group G satisfies the relation $r = t$, then it is soluble.*

In proving this theorem, the same cases are considered as in Theorem 1 of the paper ([5]); moreover, everywhere instead of π-solubility one obtains solubility, owing to the use of the fundamental theorem on π-separable groups of S. A.

Chunikhin ([12]), Lemma 1 ([4]), and arguments analogous to those given in ([10]). The theorem is also valid for the case when all prime divisors of the order of the group belong to π, i.e. when $n = 1$.

From Theorems 1 and 2 of this note and Theorem 2 ([5]) we obtain Corollary 3 ([5]), or Theorem 4 ([7]), as well as the corollaries on the solubility of πd-groups for which $r = 1$ ([4]), and on the solubility of πd-groups with $r = 2$ and $m \neq p^\alpha$, which improve Theorems 2 and 3 ([7]), or Corollaries 1 and 2 ([5]).

§ 3. Using Theorems 1 and 2, we now obtain the following results.

Theorem 3. *If G is a nonspecial πd-group and if $r > 1$ and $n > 1$, then $r \geq t$.*

Proof. Suppose the contrary, i.e. suppose that $r < t$. Since, by Lemma 3 ([4]), $r \geq t-1$, we have $r = t-1$. If $s = 0$, then, as in case 1) of Theorem 1, the collection $(*)$ would have an attainable and invariant πd-subgroup $P_{r+1}Q_1$. Then the newly formed πd-subgroup $P_1P_2Q_1$, obviously, would also be attainable and invariant in G. Since $r+1$ cannot be equal to 1 or 2, the intersection of $P_{r+1}Q_1$ with $P_1P_2Q_1$ would give, by Lemma 3 ([11]), the invariant subgroup Q_1, which is impossible. Hence $s \neq 0$.

We shall show that $s = 1$. Suppose that $s > 1$. Then form the collection of πd-subgroups $(**)$. In it, as in case 2) of Theorem 1, there is only one attainable subgroup $P_{s+1}P_1$. Since G, by Theorem 1, is soluble, it follows, in view of Hall' s theorem ([13]), that it has a subgroup $W \supset P_1$ of order $p_1^{\alpha_1}p_2^{\alpha_2}$. W, obviously, is not conjugate to any of the subgroups of the r classes of unattainable isoordinal πd-subgroups already existing in G. Therefore it is attainable in G. Since $s+1 \neq 2$, $P_{s+1}P_1 \cap W = P_1$, obviously, must be attainable in G, which contradicts the choice of it. Therefore $s = 1$; P_1 is a representative of the unique class of unattainable Sylow πd-subgroups of the group G, while $P_2, P_3, \dots, P_{r+1}$ are attainable and, consequently, invariant in G. The πd-subgroup P_2P_1 is also attainable, and in the collection of type $(**)$ one can choose r classes of unattainable πd-subgroups. Then P_3Q_1 is attainable and invariant (by Lemma 1 ([4])) in G. Form $P_1P_3Q_1$. Since it is attainable in G, the intersection $P_2P_1 \cap P_1P_3Q_1 = P_1$ will give an attainable πd-subgroup. Again we have arrived at a contradiction. Hence the relation $r = t-1$ is impossible. Therefore $r \geq t$. The theorem is proved.

Corollary. *If G is a nonspecial πd-group and if $r > 2$, then $r \geq t$.*

Obviously, here two cases are possible: $n > 1$ and $n = 1$. In the first case the corollary is valid by the theorem proved above, and in the second case it is proved analogously.

This theorem can be strengthened for $r > 3$.

Theorem 4. *If G is a nonspecial πd-group and if $r > 3$ and $n > 1$, then $r \geq t+1$.*

Proof. By Theorem 3 the relation $r < t$ is impossible. Put $r = t$. We shall first show that G contains at least one attainable Sylow πd-subgroup.

Suppose all Sylow πd-subgroups of the group G are unattainable in it. Denote one of them by P_1. Since G, by Theorem 2, is soluble, in view of Hall' s theorem ([13]) it has subgroups W_1 and W_2 of orders respectively $p_1^{\alpha_1} p_2^{\alpha_2}$ and $p_1^{\alpha_1} p_3^{\alpha_3}$. Because G has all r classes of unattainable πd-subgroups, W_1 and W_2 are attainable, which leads to the attainability of P_1. But this contradicts the supposition. Hence G has at least one Sylow attainable πd-subgroup. Further, as in Theorem 3, it could be proved that $s \neq 0$. We now show that $s = 1$. Suppose G has $s > 1$ classes of unattainable πd-subgroups and let P_1 and P_2 be representatives of two such classes. Then form the set of πd-subgroups

$$P_{s+1} P_1, \quad P_{s+2} P_1, \dots, \quad P_{rP} 1. \tag{***}$$

Consider two possibilities:

1) Suppose all πd-subgroups $(***)$ are unattainable in G. Then, obviously, the newly formed subgroup $P_{s+1} P_2$ is already attainable in G. For the same reason, the πd-subgroup W of order $p_1^{\alpha_1} p_2^{\alpha_2}$, which exists in the soluble group G by Hall' s theorem ([13]), is also attainable in G. By Lemma 3 ([11]) the intersection

$$P_{s+1} P_2 \cap W = P_2$$

is attainable in G, which is impossible. Hence in this case $s = 1$.

2) Suppose now that among the πd-subgroups $(***)$ there are some attainable in G. Then one may, obviously, take $P_{s+1} P_1$ as such a subgroup. Since in G there are already $r - 1$ classes of unattainable isoordinal πd-subgroups, the newly formed set

$$P_{s+1} P_2, \quad P_{s+2} P_2, \dots, \quad P_{rP} 2$$

will obviously consist of one attainable and one unattainable πd-subgroup. Suppose, for definiteness, that $P_{s+1} P_2$ is attainable, and $P_{s+2} P_2$ is unattainable in G. It is obvious that then

$$P_{s+1} P_2 \cap W = P_2$$

must be attainable in G, which is impossible. Consequently, also in this second case $s = 1$.

We have shown that G contains only one class of unattainable Sylow πd-subgroups. Let P_1 be a representative of this class. Form the set of πd-subgroups

$$P_1 P_2, \quad P_1 P_3, \dots, \quad P_1 P_r.$$

They are either all unattainable in G, or, by Lemma 3 ([11]), only one of them is attainable in G.

The first case leads to the presence in G of all r classes of unattainable πd-subgroups and, consequently, to the attainability in G of the πd-subgroups:

$$(P_{kP}u)P_1, \quad (P_{kP}v)P_1, \quad (P_{uP}v)P_1,$$

where k, u, v are distinct natural numbers, different from 1 and less than r. Hence it is obvious that P_1 is attainable in G. We have arrived at a contradiction.

In the second case, for definiteness, one may suppose that P_1P_2 is attainable in G. Then G will already contain $r-1$ classes of unattainable πd-subgroups. Multiplication of the attainable, and hence invariant, subgroup P_3P_4 by the unattainable P_1 will obviously give us the πd-subgroup $P_1P_3P_4$. If it is attainable in G, then the intersection

$$P_1P_2 \cap P_1P_3P_4 = P_1$$

will be an attainable subgroup in G. We have arrived at a contradiction. If, however, $P_1P_3P_4$ is unattainable in G, then, in view of the presence in G of all r classes of unattainable πd-subgroups, the newly formed πd-subgroup $P_1(P_3Q)$ is attainable in G. Then the intersection

$$P_1P_2 \cap P_1(P_3Q) = P_1$$

by Lemma 3 ([11]) will be an attainable subgroup in G.

We have obtained a contradiction. Thus the relation $r = t$ is impossible. Therefore $r \geqslant t+1$. The theorem is proved.

Corollary. *If G is a nonspecial πd-group and if $r > 4$, then $r \geqslant t+1$.*

The proof is analogous to the proof of Corollary 1.

Finally, we arrive at the following criterion:

πd-groups for which $r > 3$ when $n > 1$ (or $r > 4$ for any n) and $r < t+1$ are special.

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Note: Figure translations are in progress. See original paper for figures.