M. A. Krasnosel' skii, V. V. Strygin

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Abstract

Full Text

M. A. Krasnosel' skii, V. V. Strygin

On the Computation of the Rotation of Completely Continuous Vector Fields Connected with the Problem of Periodic Solutions of Differential Equations

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1. Consider a system of ordinary differential equations

$$\frac{dx}{dt} = f(t, x) \tag{1}$$

with right-hand side ω-periodic in t. Here x is a point of the m-dimensional space R^m, and the operator-function $f(t, x)$ is continuous in the totality of its variables.

As is easily seen, the ω-periodic solutions of equation (1) are precisely the fixed points of the operator

$$Ax(t) = x(0) - \int_t^\omega f[\tau, x(\tau)]\, d\tau, \tag{2}$$

which acts and is completely continuous in the space C of vector-functions continuous on $[0, \omega]$ (with values in R^m). The construction of the operator A is not connected with any assumptions on the smoothness of $f(t, x)$, or on the continuability of solutions of equation (1). In this connection, in a number of cases it is convenient to pass from the differential equation (1) to the equation $x = Ax$.

In order to prove theorems on the existence of periodic solutions, it is necessary to be able to compute the rotation (1) of the vector field

$$\Phi x = x - Ax \tag{3}$$

on the boundaries of domains of the space C. This problem is solved below under rather stringent assumptions on the right-hand side of equation (1). For the study of more general fields one may use (according to the standard scheme) a homotopic transition to the fields studied below.

2. Suppose, in addition, that $f(t, x)$ has such properties under which the solution $x(t) = x(t, x_0)$ of equation (1), satisfying the initial condition $x(0) = x_0$, is unique and continuable on the interval $0 \leq t \leq \omega$ (see, for example, $^{(2)}$). Put $Ux_0 = x(\omega, x_0)$. The operator U acts in R^m and is continuous.

The operator U is called the shift operator. Fixed points of the shift operator determine periodic solutions of equation (1). The proof of the existence of fixed points of the shift operator has been carried out under various conditions by a number of authors (N. N. Luzin, S. Lefschetz, U. Massera, B. P. Demidovich, N. A. Neimark, A. Halanay, and others). Almost all the theorems proved by them in fact mean that the rotation of the vector field

$$\varphi x = x - Ux \tag{4}$$

on the boundary of some domain is different from zero. The field (4) was also investigated in $^{(3)}$.

Let the field (3) have no zero vectors on the boundary Γ of a bounded domain $\Omega \subset C$. Let F be the set of fixed points of the operator A lying in Ω. By $F_0 \subset R^m$ denote the totality of values of the functions

from F at $t = 0$. Finally, let G be a bounded domain in R^m such that on its boundary Π the translation operator has no fixed points, and the set of fixed points lying in G coincides with E_0.

Theorem 1. The equality holds

$$\gamma(\Phi; \Gamma) = (-1)^m \gamma_0(\varphi, \Pi), \tag{5}$$

where $\gamma(\Phi; \Gamma)$ is the rotation of the field (3) on Γ, and $\gamma_0(\varphi; \Pi)$ is the rotation of the field (4) on Π, m is the dimension of the space R^m.

The proof consists of four basic steps.

Denote by Ω_1 the set of such vector-functions $x(t) \subset C$ that $x(0) \in \overline{G}$, and $|x(t) - x(0)| \leq M$, where M is a sufficiently large number. It turns out that the set of fixed points of the operator (2) lying in Ω_1 coincides with F. Hence the equality follows

$$\gamma(\Phi; \Gamma) = \gamma(\Phi; \Gamma_1), \tag{6}$$

where $\gamma(\Phi; \Gamma_1)$ is the rotation of the field (3) on the boundary Γ_1 of the domain Ω_1.

Put

$$T_\lambda x(t) = \begin{cases} x(t), & 0 \leq t \leq \omega - \lambda\omega, \\ y(t), & \omega - \lambda\omega \leq t \leq \omega, \end{cases} \tag{7}$$

where $y(t)$ is a solution of the system (1) satisfying the condition $y(\omega - \lambda\omega) = x(\omega - \lambda\omega)$. Consider the vector fields

$$\Phi_\lambda(x) = x - AT_\lambda x \qquad (0 \leq \lambda \leq 1). \tag{8}$$

It can be verified that the fields (8) are homotopic on Γ_1; therefore the rotations $\gamma(\Phi_0;\Gamma_1)$ and $\gamma(\Phi_1;\Gamma_1)$ are the same. The field Φ_0 coincides with the field (3). From (6) it follows that

$$\gamma(\Phi;\Gamma)=\gamma(\Phi_1;\Gamma_1). \tag{9}$$

A simple verification shows that the vectors of the field Φ_1 are not directed oppositely to the vectors of the field

$$\Phi_2 x = x(t) - 2x(0) + Ux(0).$$

Therefore the field Φ_2 is homotopic to the field Φ_1, and from (9) the equality follows

$$\gamma(\Phi;\Gamma)=\gamma(\Phi_2;\Gamma_1).$$

To compute the rotation of the field Φ_2, the theorem on the product of rotations (^{1}) may be applied. From this theorem it follows that $\gamma(\Phi_2;\Gamma_1)=\gamma_0(\Phi_2;\Pi_1)$, where $\gamma_0(\Phi_2;\Pi_1)$ is the rotation of the field Φ_2 on the intersection $\Pi_1=\Gamma_1\cap E_0$ of the boundary Γ_1 with the space E_0 of constant functions. On Π_1 the field Φ_2 has the form $\Phi_2 x = Ux - x$, and therefore $\gamma_0(\Phi_2;\Pi_1)=(-1)^m\gamma_0(\varphi;\Pi)$. Thus,

$$\gamma(\Phi;\Gamma)=\gamma(\Phi_2;\Gamma_1)=\gamma_0(\Phi_2;\Pi_1)=(-1)^m\gamma_0(\varphi;\Pi).$$

3. One can indicate a number of nonlinear operators, different from (2), whose fixed points coincide with the fixed points of the operator (2)*. We shall consider here a three-parameter family of such operators

$$Lx = ax(0)+bx(\omega)+c\int_0^\omega f[s,x(s)]\,ds+$$

$$+(1-a-b)\frac{1}{\omega}\int_0^\omega [x(s)+sf(s,x(s))]\,ds+\int_0^t f[s,x(s)]\,ds, \tag{10}$$

* V. Sh. Burd, Yu. S. Kolesov, A. Yu. Levin, and P. E. Sobolevskii took part in an interesting discussion of this question.

where a,b,c are arbitrary parameters satisfying the condition

$$\varkappa = 1-a+c\neq 0. \tag{11}$$

Let us consider the vector fields

$$\Phi(x;a,b,c)=x(t)-Lx(t). \tag{12}$$

By $\gamma(\Phi;a,b,c;\Gamma)$ we shall denote the rotation of the field (12) on the boundary Γ of the domain Ω (here and below we use the notation introduced at the beginning of § 2).

Theorem 2. If $\varkappa>0$, then

$$\gamma(\Phi; a, b, c; \Gamma) = \gamma_0(\varphi; \Pi). \tag{13}$$

If $\varkappa < 0$, then

$$\gamma(\Phi; a, b, c; \Gamma) = (-1)^m \gamma_0(\varphi; \Pi). \tag{14}$$

Equality (14) contains, in particular, the assertion of Theorem 1.

4. We shall call a continuous matrix-function $A(t)$ $(0 \leq t \leq \omega)$ regular if 1 is not an eigenvalue of the monodromy matrix [4] of the linear system $x' + A(t)x = 0$. The solutions of this system can be written in the form

$$x(t) = V(t, s)x(s). \tag{15}$$

By β below we denote the sum of the multiplicities of the real eigenvalues greater than 1 of the monodromy matrix (by the multiplicity of eigenvalues we mean the dimension of the corresponding root subspace). Let us consider the operator

$$D(x; A) = V(t,0)[I - V(\omega,0)]^{-1} \int_0^\omega V(\omega, s)[A(s)x(s) + f(s, x(s))]\, ds +$$

$$+ \int_0^t V(t, s)[A(s)x(s) + f(s, x(s))]\, ds. \tag{16}$$

This operator acts in C and is completely continuous. It is not difficult to verify that its fixed points coincide with the fixed points of the operator (2). In other words, the periodic solutions of equation (1) are the points at which the vector field

$$\Psi(A; x) = x(t) - D(x; A) \tag{17}$$

vanishes. We shall denote the rotation of this field on Γ by $\gamma(\Psi; A; \Gamma)$.

Theorem 3. The equality

$$\gamma(\Psi; A; \Gamma) = (-1)^\beta \gamma_0(\varphi; \Pi) \tag{18}$$

holds.

5. Let us indicate some simplest applications. Suppose that the right-hand side of system (1) is continuous, but does not possess such smoothness properties as guarantee uniqueness of solutions of the Cauchy problem. In this case the single-valued translation operator is not defined. The equation

$$\frac{dx}{dt} = g(t, x) \tag{19}$$

with a smooth right-hand side will be called ε-close to equation (1) if the inequality $|f(t, x) - g(t, x)| < \varepsilon$ is satisfied (in the whole space or in a sufficiently large part of it). Suppose that the solutions of all ε-close systems for $\varepsilon \leq \varepsilon_0$ are continuable to the interval $[0 \leq t \leq \omega]$ and that all these solutions are not periodic if the initial conditions belong to the boundary Π of some bounded domain $G \subset R^m$. For each ε-close system one can define the translation operator U_g, which is determined by the right-hand side $g(t, x)$ of system (19). The vector fields

$$\varphi(x; g) = x - U_{gx} \tag{20}$$

may differ substantially from one another (they may even, at some points, have oppositely directed vectors). However, the vector fields (3) corresponding to ε-close equations are homotopic to one another, and therefore Theorem 1 implies

Theorem 4. *The rotations on* Π *of the fields (20) are identical.*

This theorem, of course, can also be proved without passing to integral equations.

6. Let us now consider the autonomous system

$$\frac{dx}{dt} = f(x). \tag{21}$$

Suppose that the zero θ is an isolated singular ($f(\theta) = \theta$) point of system (21). Suppose that in some neighborhood of θ there are no periodic solutions of system (21). The Poincaré index γ_0 of the singular point θ is the rotation of the velocity field $f(x)$ on spheres of small radius centered at θ.

Theorem 5. *Let* $\gamma_0 \neq 0$. *Then the perturbed equation*

$$\frac{dx}{dt} = f(x) + \varepsilon g(t, x), \tag{22}$$

where $g(t, x)$ *is a continuous function* ω*-periodic in* t*, has periodic solutions for sufficiently small* ε*.*

This assertion follows in an obvious way from Theorem 1. Assertions analogous to Theorem 5 also follow from other considerations (see, for example, ([5],[6])).

Voronezh State University

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REFERENCES

1. M. A. Krasnosel' skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Moscow, 1956.

2. E. A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, Moscow, 1958.

3. M. A. Krasnosel' skii, A. I. Perov, DAN, 123, No. 2 (1958).

4. F. R. Gantmacher, *The Theory of Matrices*, Moscow, 1953.

5. G. I. Biryuk, DAN, 96, No. 1 (1954).

6. I. Berstein, A. Halanay, DAN, 111, No. 5 (1956).

Note: Figure translations are in progress. See original paper for figures.