



---

Soviet-era science, translated into English

# A. I. Fet

1963

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196301.66847>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

**A. I. Fet**

## **EXTREMAL PROBLEMS FOR SURFACES OF BOUNDED GAUSSIAN CURVATURE**

*(Presented by Academician S. L. Sobolev, 15 VI 1963)*

1. The influence of Gaussian curvature on the position in space of closed surfaces has been studied comparatively little. The most natural problems arise if one regards the Gaussian curvature as bounded on one side or on both sides; the investigation of such problems constituted the main content of the well-known book by W. Blaschke <sup>(1)</sup>, and since then, apparently, there has been no substantial progress in this question. We prove here some theorems on  $(n-1)$ -dimensional surfaces  $F^{n-1}$  of Euclidean  $n$ -space  $E^n$  with bounded Gaussian curvature, which have interesting applications to stability theorems in differential geometry.
2. We first prove the following property of convex polyhedra in  $E^n$ .

**Theorem 1.** *Let a one-to-one correspondence be established between the faces of the convex polyhedra  $P_1^{n-1}$ ,  $P_2^{n-1}$ , with corresponding faces having identically directed exterior normals. Then, if the area  $F_i^{(2)}$  of each face of  $P_2^{n-1}$  is not less than the area  $F_i^{(1)}$  of the corresponding face of  $P_1^{n-1}$ , the volume of  $P_2^{n-1}$  is not less than the volume of  $P_1^{n-1}$ , and equality is possible only in the case of congruence of  $P_1^{n-1}$  and  $P_2^{n-1}$ .*

**Proof.** Denote by  $h_i$  the support numbers, and by  $F_i$  the areas of the faces of a convex polyhedron  $P^{n-1}$  with the same directions of exterior normals as those of  $P_1^{n-1}$ ; let  $V(P^{n-1})$  be the volume of  $P^{n-1}$ . Then, as Minkowski proved, the polyhedron homothetic to  $P_1^{n-1}$  realizes the maximum of volume in the class of all convex polyhedra  $P^{n-1}$  with the same directions of exterior normals, satisfying the condition

$$\sum h_i F_i^{(1)} = 1$$

(see, for example, <sup>(2)</sup>, pp. 292-295; the generalization of the proof to the  $n$ -dimensional case is obvious).

Hence

$$\frac{1}{n} [V(P_1^{n-1})]^{-(n-1)/n} = \frac{[V(P_1^{n-1})]^{1/n}}{\sum h_i^{(1)} F_i^{(1)}} \geq \frac{[V(P_2^{n-1})]^{1/n}}{\sum h_i^{(2)} F_i^{(1)}} \geq$$

$$\geq \frac{[V(P_2^{n-1})]^{1/n}}{\sum h_i^{(2)} F_i^{(2)}} = \frac{1}{n} [V(P_2^{n-1})]^{-(n-1)/n},$$

therefore,  $V(P_1^{n-1}) \leq V(P_2^{n-1})$ . The equality sign is possible only if  $F_i^{(1)} = F_i^{(2)}$  for all  $i$ ; but then, by another theorem of Minkowski,  $P_1^{n-1}$  and  $P_2^{n-1}$  are congruent (see (2), Ch. 8, § 3).

3. Let two arbitrary closed convex surfaces  $\Gamma_1^{n-1}, \Gamma_2^{n-1}$  in  $E^n$  be given with surface functions  $F_1(M),$  respectively  $F_2(M),$  where  $M$  are Borel sets of the unit sphere  $S^{n-1}$  (3). If

for all  $M, F_2(M) \geq F_1(M),$  we shall say that the Gaussian curvature of  $\Gamma_2^{n-1}$  at each point is no greater than the Gaussian curvature of  $\Gamma_1^{n-1}$  at the point with the same exterior normal.

**Theorem 2.** *Let  $\Gamma_1^{n-1}, \Gamma_2^{n-1}$  be arbitrary convex surfaces, and let the Gaussian curvature of  $\Gamma_2^{n-1}$  at each point be no greater than the Gaussian curvature of  $\Gamma_1^{n-1}$  at the point with the same exterior normal. Then the volume of the body bounded by  $\Gamma_2^{n-1}$  is not less than the volume of the body bounded by  $\Gamma_1^{n-1},$  and equality is possible only in the case when  $\Gamma_1^{n-1}$  and  $\Gamma_2^{n-1}$  are congruent.*

The proof is based on approximating the surface functions of  $\Gamma_2^{n-1}$  and  $\Gamma_1^{n-1}$  by surface functions  $\hat{F}_2(M) \geq \hat{F}_1(M)$  consisting of a finite number of point loads, with the loads for both functions concentrated at the same points; then  $\hat{F}_1, \hat{F}_2$  are realized by convex polyhedra, Theorem 1 is applied, and finally A. D. Aleksandrov's uniqueness theorem is used in the passage to the limit.

Consequences of Theorem 2 are the extremal properties of the sphere stated in the following theorem; note that the second of them also follows from simpler considerations.

**Theorem 3.** *Among all convex surfaces whose Gaussian curvature is no greater than (respectively, no less than) a given  $k_0,$  the sphere, and only the sphere, bounds the least (greatest) volume.*

4. We shall now prove  $n$ -dimensional generalizations of the well-known theorems of Blaschke on the symmetrization of convex surfaces ((16), pp. 123-126, 141-142).

**Theorem 4.** *If the Gaussian curvature of a convex surface  $\Gamma^{n-1}$  is not less than  $k_0,$  then the surface  $S(\Gamma^{n-1})$  obtained from  $\Gamma^{n-1}$  by Steiner symmetrization also has curvature not less than  $k_0.$*

It is obviously enough to carry out the proof in the case  $k_0 > 0.$

**Lemma.** *Let the matrices of order  $m,$   $\|a_{ij}\|, \|b_{ij}\|,$  be positive definite; then  $[\det |(1-t)a_{ij} + tb_{ij}|]^{1/m}$  ( $0 \leq t \leq 1$ ) is a concave function of  $t.$*

Since forms with matrices  $a_{ij}, b_{ij}$  can be simultaneously reduced to orthogonal form, these matrices may be assumed diagonal; then the lemma follows from the known inequality

$$\left[ \prod_{i=1}^m (a_i + b_i) \right]^{1/m} \geq \left[ \prod_{i=1}^m a_i \right]^{1/m} + \left[ \prod_{i=1}^m b_i \right]^{1/m}, \quad a_i \geq 0, b_i \geq 0$$

(see <sup>(5)</sup>, vol. 1, sec. 2.84).

The principal curvatures  $k_i$  of the surface  $\Gamma^{n-1}$  satisfy the equation  $\det |b_{ij} - kg_{ij}| = 0$ , where  $g_{ij}, b_{ij}$  are the coefficients of the first and second quadratic forms of  $\Gamma^{n-1}$  (<sup>(6)</sup>, p. 238). Represent a part of the surface  $\Gamma^{n-1}$  in the form  $z = f(x^1, \dots, x^{n-1})$ , where  $x^1, \dots, x^{n-1}$  are Cartesian coordinates in the plane with respect to which the symmetrization is performed, and the  $z$ -axis is orthogonal to this plane. Leaving aside the simpler case when the normal to  $\Gamma^{n-1}$  is perpendicular to the  $z$ -axis, we have  $g_{ij} = \delta_{ij} + p_i p_j$ ,

$$b_{ij} = \Delta^{-(n-1)/2} q_{ij}, \quad \text{where } p_i = \frac{\partial f}{\partial x^i}, \quad q_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}, \quad \Delta = 1 + \sum_{i=1}^{n-1} p_i^2,$$

whence the Gaussian curvature

$$k = \prod_{i=1}^{n-1} k_i = \det |b_{ij}| [\det |g_{ij}|]^{-1} = \det |q_{ij}| (\sqrt{\Delta})^{-(n+1)}.$$

Let  $\Gamma_1^{n-1}, \Gamma_2^{n-1}, \Gamma_3^{n-1}$  be convex surfaces of the form, respectively,  $z = f_1(x^1, \dots, x^{n-1}), z = f_2(x^1, \dots, x^{n-1}), z = \frac{1}{2}(f_1 + f_2)$ . Put

$$p_i^{(l)} = \frac{\partial f_l}{\partial x^i}, \quad q_{ij}^{(l)} = \frac{\partial^2 f_l}{\partial x^i \partial x^j}, \quad \varphi(t) = \left[ \det \left[ (1-t)q_{ij}^{(1)} + tq_{ij}^{(2)} \right] \right]^{1/(n-1)},$$

$$\psi(t) = \left\{ 1 + \sum_{i=1}^{n-1} \left[ (1-t)p_i^{(1)} + tp_i^{(2)} \right]^2 \right\}^{(n+1)/2(n-1)} = \left( \sqrt{\Delta(t)} \right)^{(n+1)/(n-1)},$$

$$\chi(t) = \frac{\varphi(t)}{\psi(t)}.$$

Then  $[\chi(t)]^{n-1}$  for  $t = 0, 1, \frac{1}{2}$  is equal, respectively, to the Gaussian curvature of  $\Gamma_1^{n-1}, \Gamma_2^{n-1}, \Gamma_3^{n-1}$  at the point with coordinates  $x^1, \dots, x^{n-1}$ . Therefore, to prove

the theorem it is sufficient to verify that  $\chi(t) \geq \eta(t)$ , where  $\eta(t)$  is monotone and  $[\eta(0)]^{n-1}$ ,  $[\eta(1)]^{n-1}$  are equal to the Gaussian curvatures of  $\Gamma_1^{n-1}$ ,  $\Gamma_2^{n-1}$ . Put

$$\varphi_0(t) = (1-t)\varphi(0) + t\varphi(1), \quad \psi_0(t) = (1-t)\psi(0) + t\psi(1),$$

$$\eta(t) = \frac{\varphi_0(t)}{\psi_0(t)}.$$

By the lemma,  $\varphi(t) \geq \varphi_0(t)$ . Since  $\Delta(t) > 0$ , the equation  $y^2 = \Delta(t)$  determines in the  $(t, y)$ -plane a hyperbola or a pair of straight lines; hence  $\sqrt{\Delta(t)}$ , and consequently also  $\psi(t)$ , is convex on  $[0, 1]$ , and  $\psi(t) \leq \psi_0(t)$ . Finally,  $\eta(t)$  is monotone for  $0 \leq t \leq 1$ , which proves the theorem.

**Theorem 5.** *If the Gaussian curvature of a convex surface  $\Gamma^{n-1}$  is not greater than  $k_0$ , then the surface  $M(\Gamma^{n-1})$ , obtained from  $\Gamma^{n-1}$  by Minkowski symmetrization ((1), p. 103), also has curvature not greater than  $k_0$ .*

The proof is based on the expression of the principal radii of curvature in terms of the support function ((4), p. 61) and on the lemma given above.

5. The preceding theorems make it possible to transfer to the  $n$ -dimensional case the methods of solving extremal problems used by Blaschke. It will be convenient for us to formulate these methods at once for a certain class of functionals.

We shall call a functional  $T(\Gamma^{n-1})$ , defined on some set  $X$  of convex surfaces  $\Gamma^{n-1}$ , convex with respect to Steiner symmetrization if, together with  $\Gamma^{n-1}$ ,  $X$  contains  $S(\Gamma^{n-1})$  and  $T(S(\Gamma^{n-1})) \leq T(\Gamma^{n-1})$ , whatever the Steiner symmetrization  $S$ . Functionals convex with respect to Minkowski symmetrization are defined analogously.

**Theorem 6.** 1) *A functional convex with respect to Steiner symmetrization, among all convex surfaces of curvature not less than  $k_0$ , assumes its least value on the sphere.*

- 2) *A functional convex with respect to Minkowski symmetrization, among all convex surfaces of curvature not greater than  $k_0$ , assumes its least value on the sphere.*

The proof is based on Theorems 4, 5; the corresponding theorems are also true for concave functionals.

A number of known estimates follow from Theorem 6. We note separately two consequences. Let the point  $O$  lie inside  $\Gamma^{n-1}$ , and let the Gaussian curvature of  $\Gamma^{n-1}$  be not greater than  $k_0 = R^{-(n-1)}$ ; denote by  $r(n)$  the distance from  $O$  to the point of  $\Gamma^{n-1}$  with exterior normal  $n$ . Then, by Theorem 6, 2),

$$\left\{ \int_{S^{n-1}} [r(n)]^p dn \right\}^{1/p} \geq \omega_{n-1} R, \quad p \geq 1,$$

where  $\omega_{n-1}$  is the area of  $S^{n-1}$ . Choosing  $O$  at the midpoint of the diameter  $D$ , we obtain, as  $p \rightarrow \infty$ , the estimate mentioned in the literature,  $D \geq 2R$ . Since the surface element  $dF \geq \frac{1}{k_0} dn$ , we further have the inequality

$$\left[ \int_{\Gamma^{n-1}} r^p dF \right]^{1/p} \geq R^n.$$

The volume  $V(\Gamma^{n-1})$  is not a convex functional; nevertheless, from Theorem 6, 2), as V. I. Diskant has observed, the first assertion of Theorem 3 can be derived.

For this it suffices to take, as  $T$ , the integral of the support function  $\int_{S^{n-1}} H dn$

and to use the inequality  $dE \geq \frac{1}{k_0} dn$ .

Institute of Mathematics with Computing Center  
Siberian Branch of the Academy of Sciences of the USSR

Received  
7 VI 1963

## REFERENCES

1. W. Blaschke, *Kreis und Kugel*, a) Leipzig, 1916; b) Berlin, 1956.
2. A. D. Aleksandrov, *Convex Polyhedra*, 1950.
3. A. D. Aleksandrov, *Mat. sbornik*, **6** (48), 167 (1939).
4. T. Bonnesen, W. Fenchel, *Theorie der konvexen Körper*, Berlin, 1934.
5. G. Pólya, G. Szegő, *Problems and Theorems in Analysis*, 2nd ed., Moscow, 1956.
6. L. P. Eisenhart, *Riemannian Geometry*, Moscow, 1948.

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*