MATHEMATICS

I. V. OSTROVSKII

1963

SovietRxiv

Abstract

Full Text

MATHEMATICS

I. V. OSTROVSKII

ON A PROBLEM IN THE THEORY OF VALUE DISTRIBUTION

(Presented by Academician S. N. Bernstein, 29 I 1963)

1°. Let $f(z)$ be a meromorphic function; $n(r,a)$, $N(r,a)$, $m(r,f)$, $T(r)$ are the quantities used in Nevanlinna theory that characterize this function. Put

$$n(r) = n(r,0) + n(r,\infty); \qquad N(r) = N(r,0) + N(r,\infty);$$

$$\varkappa(f) = \varliminf_{r\to\infty} N(r)[T(r)]^{-1}.$$

We shall denote by ρ the order of the function $f(z)$, and by λ its lower order.

R. Nevanlinna proved ([3]) that

$$\varkappa(f) \geqslant k(\rho),$$

where $k(\rho)$ is a quantity positive for nonintegral ρ, and posed the problem of finding the best lower estimate for $\varkappa(f)$ in terms of ρ.

The first, though rather crude, estimate of $\varkappa(f)$ for $0 \leqslant \rho \leqslant 1$ was obtained by Shah ([7]), and a substantially more precise one by A. A. Gol' dberg ([2]). Recently Edrei and Fuchs ([4,5]) found the best estimate of $\varkappa(f)$ for $0 \leqslant \rho \leqslant 1$ and an estimate of $\varkappa(f)$ for $\rho > 1$ fairly close to the best possible. The result of Edrei and Fuchs may be formulated as follows:

Theorem A. Put

$$\nu(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 0.5, \\ \sin \pi x, & 0.5 \leqslant x \leqslant 1, \\ |\sin \pi x| \{2.2x + 0.5|\sin \pi x|\}^{-1}, & 1 \leqslant x \leqslant \infty. \end{cases} \tag{1}$$

The estimate

$$\varkappa(f) \geqslant \nu(\rho) \tag{2}$$

holds.

The main result of this article is the following theorem, which generalizes Theorem A.

Theorem 1. Let $\nu(x)$ be defined by relation (1). The estimate

$$\varkappa(f) \geqslant \max_{\lambda \leqslant x \leqslant \rho} \nu(x) \tag{3}$$

holds.

2°. In this section we shall give two assertions (Lemmas 1 and 2) used in the proof of Theorem 1. These assertions are due to Edrei and Fuchs ([5]).

Lemma 1. Let $f(z)$, $f(0)=1$, be a meromorphic function, and let $q \geqslant 0$ be an integer. For any $R>0$ and $0 \leqslant r \leqslant 0.5R$, the relation*

$$2T(r) - N(r) \leqslant r^q \int_0^R n(t) t^{-q-1} \Phi\left(\frac{t}{r}\right) dt + K_1 \left(\frac{r}{R}\right)^{q+1} T(2R) + K_2 r^q, \tag{4}$$

holds, where

$$\Phi(t) = \frac{1}{2\pi} \int_0^{2\pi} |te^{i\theta} - 1|^{-1} \, d\theta.$$

* We shall agree to denote by the letter K with indices positive quantities not depending on the variables denoted by the letters R, r, t, s, z.

Lemma 1 is contained in Theorem 3b of the paper ([5]). Since the proof of this theorem is very cumbersome, we shall give here a comparatively simple argument which makes it possible to obtain Lemma 1.

We shall rely on the following assertion, contained implicitly in the work of Ph. Nevanlinna ([6]) (see also ([1]), p. 225).

Let $f(z)$, $f(0)=1$, be a meromorphic function with zeros a_μ and poles b_ν, and let $q \geq 0$ be an integer. For every $R>0$ the relation

$$f(z) = \alpha_R(z) \omega_R(z),$$

holds, where

$$\alpha_R(z) = \prod_{|a_\mu|<R} E\left(\frac{z}{a_\mu}, q\right) \left\{ \prod_{|b_\nu|<R} E\left(\frac{z}{b_\nu}, q\right) \right\}^{-1}$$

($E(z,q)$ is the canonical Weierstrass factor of genus q), and $\ln\omega_R(z)$ for $|z| \leq 0.5R$ ($\ln\omega_R(0) = 0$) admits the estimate

$$|\ln\omega_R(z)| \leq K_3(rR^{-1})^{q+1}T(2R) + K_4 r^q \qquad (|z| = r \leq 0.5R). \tag{5}$$

To obtain this assertion, it suffices to integrate $q+1$ times with respect to z, from 0 to z, the expression for $\{\ln f(z)\}^{(q+1)}$ given in the book ([1]) (p. 225), and to take into account the estimates for $S_R(z)$ and $I_R(z)$.

We borrow the subsequent arguments from Edrei and Fuchs ([5]). Integrating with respect to θ from 0 to 2π the known inequality

$$\left|\ln\left|E(re^{i\theta},q)\right|\right| \leq \int_0^r s^q\left|se^{i\theta}-1\right|^{-1}\,ds,$$

we obtain the estimate

$$m(r,E(r,q)) + m(r,\{E(z,q)\}^{-1}) \leq \int_0^r s^q\Phi(s)\,ds = \int_1^\infty s^{-q-1}\Phi\left(\frac{s}{r}\right)\,ds.$$

Using this estimate, we have

$$m(r,\alpha_R) + m(r,\alpha_R^{-1}) \leq r^q\int_0^R\left\{\int_u^\infty t^{-q-1}\Phi\left(\frac{t}{r}\right)\,dt\right\}dn(u) =$$

$$= r^q\left\{\int_0^R n(t)t^{-q-1}\Phi\left(\frac{t}{r}\right)\,dt + n(R)\int_R^\infty t^{-q-1}\Phi\left(\frac{t}{r}\right)\,dt\right\} \leq$$

$$\leq r^q\int_0^R n(t)t^{-q-1}\Phi\left(\frac{t}{r}\right)\,dt + K_5\left(\frac{r}{R}\right)^{q+1}T(2R) \qquad (0 \leq r \leq 0.5R).$$

But, by virtue of (5),

$$m(r,\omega_R) + m(r,\omega_R^{-1}) \leq K_6\left(\frac{r}{R}\right)^{q+1}T(2R) + K_7 r^q \qquad (0 \leq r \leq 0.5R).$$

It remains to note that

$$2T(r) - N(r) = m(r,f) + m(r,f^{-1}) \leq$$

$$\leq m(r, \alpha_R) + m(r, \alpha_R^{-1}) + m(r, \omega_R) + m(r, \omega_R^{-1}).$$

Lemma 2. Put

$$J(\beta) = \int_0^\infty t^{-\beta-1}\Phi(t^{-1})\,dt, \qquad 0 < \beta < 1.$$

The estimate is valid

$$J(\beta) \leq 4.4 \operatorname{cosec} \pi\beta. \tag{6}$$

We note that estimate (6) is obtained from the relation established by Edrei and Fuchs (5) *

$$J(\beta) = \pi^2 \operatorname{cosec} \pi\beta \left\{ \Gamma\left(1 - \frac{\beta}{2}\right) \Gamma\left(\frac{1+\beta}{2}\right) \right\}^{-2}. \tag{7}$$

(the expression in braces attains its minimum at $\beta = 0.5$).

3°. **Proof of Theorem 1.** Let us first make a number of remarks simplifying our problem.

We may assume that $\lambda < \rho$, for when $\lambda = \rho$, estimate (3) coincides with (2). Since the function $\nu(x)$ is continuous, it suffices to prove that for every nonintegral x satisfying the condition $\lambda < x < \rho$, the relation

$$\varkappa(f) \geqslant \nu(x) \tag{8}$$

holds.

In the note (9) the author proved that, for $\lambda < 1$, $\varkappa(f) \geqslant \nu(\lambda)$ holds. Since $\nu(x)$ is nonincreasing for $0 \leqslant x \leqslant 1$, it follows that (8) holds for $\lambda < x < \min(\rho, 1)$ (and hence Theorem 1 also holds when $\rho \leqslant 1$). Thus our problem reduces to proving the validity of relation (8) for all nonintegral values x satisfying the condition $\max(\lambda, 1) < x < \rho$ (> 1).

Without loss of generality, one may assume that $f(0) = 1$. Put $q = [x]$ in relation (4). Let $\varkappa'$ be an arbitrary number satisfying $\varkappa' > \varkappa(f)$. Then, for $r \geqslant r_0$, we shall have $N(r) \leqslant \varkappa' T(r)$, and hence

$$(2 - \varkappa')T(r) \leqslant r^q \int_0^R n(t)t^{-q-1}\Phi\left(\frac{t}{r}\right)dt + K_8\left(\frac{r}{R}\right)^{q+1} T(2R) + K_9 r^q$$

$$(r_0 \leqslant r \leqslant 0.5R).$$

Multiplying both sides of this relation by r^{-x-1} and integrating with respect to r from r_0 to $0.5R$, we obtain

$$(2-\varkappa')\int_{r_0}^{0.5R} T(r)r^{-x-1}\,dr \leqslant$$

$$\leqslant \int_{r_0}^{0.5R} r^{q-x-1}\left\{\int_0^R n(t)t^{-q-1}\Phi\left(\frac{t}{r}\right)\,dt\right\}\,dr + K_{10}T(2R)R^{-x} + K_{11}. \quad (9)$$

Denote the repeated integral standing on the right-hand side by I. We have

$$I = \int_0^R n(t)t^{-q-1}\left\{\int_{r_0}^{0.5R} r^{q-x-1}\Phi\left(\frac{t}{r}\right)\,dr\right\}\,dt =$$

$$= \int_0^R n(t)t^{-x-1}\left\{\int_{r_0/t}^{0.5R/t} u^{q-x-1}\Phi(u^{-1})\,du\right\}\,dt \leqslant \int_0^R n(t)t^{-x-1}\left\{\int_0^\infty u^{q-x-1}\Phi(u^{-1})\,du\right\}\,dt =$$

$$= J(x-q)\int_0^R n(t)t^{-x-1}\,dt = J(x-q)\left\{x\int_0^R N(t)t^{-x-1}\,dt + N(R)R^{-x}\right\} =$$

* Relation (7) is also obtained in the following way. $J(\beta)$ can be represented in the form

$$J(\beta) = \frac{1}{\pi}\int_0^1 (t^{\beta-1}+t^{-\beta})\left\{\int_0^\pi (t^2+1-2t\cos\theta)^{-0.5}\,d\theta\right\}\,dt.$$

Then, expanding $(t^2+1-2t\cos\theta)^{-0.5}$ in a series in Legendre polynomials $P_n(\cos\theta)$ and carrying out the integration, we find for $J(\beta)$ a representation in the form of a series. This series (see (8), vol. 1, p. 8, formula (4)) is the Mittag-Leffler series of the function

$$\frac{\Gamma\left(\frac{1-\beta}{2}\right)\Gamma\left(\frac{\beta}{2}\right)}{2\Gamma\left(1-\frac{\beta}{2}\right)\Gamma\left(\frac{1+\beta}{2}\right)},$$

which coincides with the right-hand side of (7).

$$= J(x{-}q)\left\{x\int_{r_0}^{0.5R} N(t)t^{-x-1}\,dt + x\int_0^{r_0} N(t)t^{-x-1}\,dt + x\int_{0.5R}^{R} N(t)t^{-x-1}\,dt+\right.$$

$$\left.+N(R)R^{-x}\right\} \leqslant J(x-q)xx'\int_{r_0}^{0.5R} T(t)t^{-x-1}\,dt + K_{12}T(R)R^{-x} + K_{13}.$$

Substituting this estimate into (9), we obtain the inequality

$$(2 - x' - J(x-q)xx')\int_{r_0}^{0.5R} T(t)t^{-x-1}\,dt \leqslant K_{14}T(2R)R^{-x} + K_{15}. \qquad (10)$$

Since $x < \rho$, as $R \to \infty$

$$\int_{r_0}^{0.5R} T(t)t^{-x-1}\,dt \to \infty.$$

Since $x > \lambda$, there is a sequence $R_k \uparrow \infty$ such that

$$T(2R_k)R_k^{-x} \to 0 \qquad (k \to \infty).$$

Consequently, letting $R \to \infty$ in (10) along this sequence R_k, we conclude that

$$2 - x' - J(x-q)xx' \leqslant 0,$$

whence

$$x' \geqslant 2\{1 + xJ(x-q)\}^{-1}.$$

Since the last relation holds for any $x' > \varkappa(f)$, it also holds for $x' = \varkappa(f)$. To obtain (8), it remains to use relation (6) with $\beta = x - q$.

4°. Theorem 1 makes it possible to shorten substantially the path to one important result (Theorem 6 of paper ([5])), which is obtained in a strengthened form.

Theorem 2. *Let $f(z)$ be a meromorphic function of finite lower order λ, and let p be an integer determined by the condition*

$$p - 0.5 \leqslant \lambda < p + 0.5.$$

If

$$\varkappa(f) < \beta(5e(p+1))^{-1}, \qquad 0 < \beta \leqslant 5, \tag{11}$$

then $p \geqslant 1$ *and*

$$p - 0.1\beta < \lambda \leqslant p < p + 0.1\beta.$$

In Theorem 6 of paper ([5]) it is asserted that, when condition (11) is fulfilled with $0 < \beta \leqslant 0.5$, the relations

$$|\rho - p| < 0.1\gamma, \qquad p - \beta \leqslant \lambda < p + 0.1\beta$$

hold.

To obtain Theorem 2, it suffices to note that the set $Q_p = \{x : \nu(x) < \beta(5e(p+1))^{-1}\}$ contains no points of the interval $\{x : 0 \leqslant x \leqslant 0.5\}$, and to estimate the length of the interval $Q_p \cap \{x : p - 0.5 \leqslant x \leqslant p + 0.5\}$.

I express my gratitude to A. A. Goldberg for his attention to the work and for valuable comments.

Kharkov State University
named after A. M. Gorky

Received
24 I 1963

References

[1] R. Nevanlinna, *Single-Valued Analytic Functions*, Moscow–Leningrad, 1941.
[2] A. A. Goldberg, DAN, **114**, No. 2, 245 (1957).
[3] R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Paris, 1929.
[4] A. Edrei, W. H. J. Fuchs, Duke Math. J., **27**, No. 2, 233 (1960).
[5] A. Edrei, W. H. J. Fuchs, Trans. Am. Math. Soc., **93**, No. 2, 292 (1959).
[6] F. Nevanlinna, Soc. Sci. Fenn. Comm. Phys.-Math., **2**, No. 4, 1 (1923).
[7] S. M. Shah, Math. Student, **12**, 67 (1944).
[8] H. Bateman, *Higher Transcendental Functions*, N. Y., 1953.
[9] I. V. Ostrovskii, DAN, **150**, No. 1 (1963).

Note: Figure translations are in progress. See original paper for figures.