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Abstract

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PHYSICS

A. A. BOGUSH

WAVE FUNCTIONS AND CALCULATION OF MATRIX ELEMENTS FOR LONGITUDINALLY POLARIZED VECTOR PARTICLES

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Recently, in connection with the experimental discovery of particles with spin 1 (see, for example, [1,2] and the bibliography cited there), the development of the theory of vector mesons has acquired great interest.

Below, within the framework of the 10-dimensional β-formalism, a compact matrix representation of the wave functions of a vector meson is proposed, allowing calculations of various interaction effects involving longitudinally polarized particles with spin 1 to be carried out on the basis of a covariant method of direct evaluation of matrix elements.

In accordance with the general method of projection operators in the theory of elementary particles [3], the projection matrices-dyads $\Lambda_p^{(s)}$, which determine the states of a vector particle with a given value of the 4-momentum and of the projection of the spin on the direction of motion, can be written in the form*

$$\Lambda_p^{(\pm 1)} = -\frac{1}{4\chi^2}\sigma_s(\sigma_s \pm 1)\hat{p}(\hat{p} - i\chi) = \psi^{(\pm 1)}(p) \cdot \bar{\psi}^{(\pm 1)}(p); \tag{1}$$

$$\Lambda_p^{(0)} = \frac{1}{2\chi^2}(\sigma_s^2 - 1)\hat{p}(\hat{p} - i\chi) = \psi^{(0)}(p) \cdot \bar{\psi}^{(0)}(p). \tag{2}$$

Here $\hat{p} = p_\mu \beta_\mu$, where β_μ are the 10×10 Duffin–Kemmer matrices;

$$\sigma_s = i\delta_{abc} s_a \beta_b \beta_c = s\sigma \qquad \left(s = -\frac{\mathbf{p}}{|\mathbf{p}|}\right) \tag{3}$$

is the operator of spin projection on the direction of motion of the particle; $\psi^{(\pm 1)}(p)$, $\psi^{(0)}(p)$ are 10-dimensional wave functions describing the states of a vector particle with spin projections ± 1 and 0, respectively.

Just as in the 10-dimensional theory of the electromagnetic field (see [4–6]), the spin-projection operator can be expressed through the mutually orthogonal unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ ($[\mathbf{e}_1\mathbf{e}_2] = \mathbf{s}$) in the form

$$\sigma_s = i(\hat{e}_1\hat{e}_2 - \hat{e}_2\hat{e}_1), \qquad \hat{e} = e_a\beta_a, \tag{4}$$

or through the circular vectors $\mathbf{e}^{(+)}$ and $\mathbf{e}^{(-)}$:

$$\mathbf{e}^{(\pm)} = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \pm i\mathbf{e}_2), \qquad \mathbf{e}^{(+)}\mathbf{e}^{(-)} = 1, \qquad \mathbf{e}^{(\pm)2} = 0 \tag{5}$$

in the form

$$\sigma_s = \hat{e}^{(-)}\hat{e}^{(+)} - \hat{e}^{(+)}\hat{e}^{(-)}. \tag{6}$$

Such a representation is convenient for describing spin states with spin projection ± 1. According to (6), taking into account the Duffin–Kemmer algebra for the projection spin operators $\beta_{(\pm 1)}$ [3], which select from the wave–

* We restrict ourselves to the case of positive energies.

of the wave function of a state with spin projections $+1$ and -1, we shall have

$$\beta_{(+1)} = \frac{1}{2}\sigma_s(\sigma_s + 1) = (\hat{e}^{(-)})^2(\hat{e}^{(+)})^2; \tag{7}$$

$$\beta_{(-1)} = \frac{1}{2}\sigma_s(\sigma_s - 1) = (\hat{e}^{(+)})^2(\hat{e}^{(-)})^2. \tag{8}$$

In this case the projection matrices-dyads $\Lambda_p^{(\pm 1)}$ (1) take the form

$$\Lambda_p^{(\pm 1)} = -\frac{1}{2\chi^2}(\hat{e}^{(\mp)})^2(\hat{e}^{(\pm)})^2\hat{p}\,(\hat{p} - i\chi). \tag{9}$$

Using the method of covariantization described in [7], the spin-projection operators (3), (4), (6), in complete analogy with how this was done for the electromagnetic field in [8], can be put into the relativistically covariant form

$$\sigma_{s'} = \frac{1}{\chi}\delta_{\mu\nu\rho\sigma}p'_\mu s'_\nu\beta_\rho\beta_\sigma = \frac{1}{\chi}\{(s'_0\mathbf{p}' - p'_0\mathbf{s}')\sigma + i[\mathbf{p}'\mathbf{s}']\tau\}; \tag{10}$$

$$\sigma_{s'} = i(\hat{e}'_1\hat{e}'_2 - \hat{e}'_2\hat{e}'_1); \tag{11}$$

$$\sigma_{s'} = \hat{e}'^{(-)}\hat{e}'^{(+)} - \hat{e}'^{(+)}\hat{e}'^{(-)}, \tag{12}$$

where $\hat{e}' = \hat{e}_\mu \beta_\mu$, $e' = Le$, $e = \binom{\mathbf{e}}{0}$, $p' = Lp$, $s' = Ls$, $s = \binom{\mathbf{s}}{0}$, $\sigma_a = i\delta_{abc}\beta_b\beta_c$, $\tau_a = i(\beta_a\beta_\mu - \beta_\mu\beta_a)$ $(a, b, c = 1, 2, 3)$, and L is the Lorentz matrix corresponding to a pure motion.

Naturally, under this the projection matrices $\Lambda_p^{(s)}$ (1), (2) retain their general structure. Thus, for example, expression (9) takes the form

$$\Lambda_{p'}^{(\pm 1)} = -\frac{1}{2\chi^2}(\hat{e}'^{(\mp)})^2(\hat{e}'^{(\pm)})^2\hat{p}'\,(\hat{p}' - i\chi). \tag{13}$$

Proceeding analogously to [8], i.e., using the representation of the matrices

$$\beta_\mu = \varepsilon^{\mu\rho|\rho} + \varepsilon^{\rho[\mu\rho]} \tag{14}$$

in terms of the elements of the complete matrix algebra ε^{AB} in 10-dimensional space ($\varepsilon^{AB}\varepsilon^{CD} = \delta_{BC}\varepsilon^{AD}$, where $A, B, C, D = \rho, [\mu\nu]$, $\mu \neq \nu$), after the corresponding calculations we find *

$$\psi^{(\pm 1)}(p) = \frac{1}{\chi\sqrt{2}}(\hat{p} - i\chi)e_\mu^{(\mp)}\varepsilon^{\mu 1}. \tag{15}$$

Here it has been taken into account that

$$\bar{\psi} = \psi^*\eta, \qquad \eta = 2\beta_4^2 - 1, \qquad \eta^2 = 1, \qquad e^{(+)*} = e^{(-)}, \qquad e_4^{(+)*} = -e_4^{(-)}, \tag{16}$$

and, consequently,

$$(\hat{p} - i\chi)\eta = -\eta(\hat{p} - i\chi)^*, \qquad \varepsilon^{ik}\eta = \varepsilon^{ik}, \qquad \varepsilon^{i4}\eta = -\varepsilon^{i4}. \tag{17}$$

If the wave functions (15) are written out in expanded form, we obtain

$$\psi^{(\pm 1)}(p) = \frac{1}{\chi\sqrt{2}}\begin{vmatrix} -i\chi e^{(\mp)} \\ -i\chi e_4^{(\mp)} \\ [\mathbf{p}\, e^{(\mp)}] \\ -(p_4 e^{(\mp)} - e_4^{(\mp)}\mathbf{p}) \end{vmatrix}. \tag{18}$$

As a special case of (18) there follow formulas (8) of paper [9] (for $s = \pm 1$).

* For definiteness, the column different from zero has been taken to be the first.

Applying the covariant method of calculating [3] the squared modulus of the matrix element, one can write

$$\left|M_{p_1\to p_2}^{s_1\to s_2}\right|^2 = \mathrm{Sp}\left\{Q\Lambda_{p_1}^{(s_1)}\overline{Q}\Lambda_{p_2}^{(s_2)}\right\}, \tag{19}$$

where Q is the vertex operator, and $\Lambda_{p_1}^{(s_1)}$ and $\Lambda_{p_2}^{(s_2)}$ are the projection dyad matrices (1), (2), determining the initial and final states of the vector particle. Such an approach proves especially convenient when considering processes involving unpolarized vector particles, since in this case the summation and averaging over spin states can be performed before taking traces. As a result we obtain

$$\left|M_{p_1\to p_2}\right|^2 = \frac{2}{(2\varkappa^2)^2}\,\mathrm{Sp}\left\{Q\hat{p}_1(\hat{p}_1 - i\varkappa)\overline{Q}\hat{p}_2(\hat{p}_2 - i\varkappa)\right\}. \tag{20}$$

In calculations of interaction processes of particles with fixed initial and final polarizations, especially when the vertex operator has a complicated form, the computations nevertheless turn out to be rather cumbersome. In this case, considerable simplifications and shortening of the calculations can be achieved by directly computing the matrix elements of polarized particles. In contrast to the method used in [9] and based on direct multiplication of wave functions and matrices β_μ, taken explicitly in some chosen basis, we use the covariant approach. The matrix element is written in the form [7]

$$M_{p_1\to p_2}^{s_1\to s_2} = \mathrm{Sp}\left\{Q\psi^{(s_1)}(p_1)\cdot\overline{\psi}^{(s_2)}(p_2)\right\}. \tag{21}$$

In order, when calculating the matrix element, to be able to use the technique of traces, it is necessary to express the matrix dyad $\psi_1 \cdot \overline{\psi}_2$ in terms of the 10×10 Duffin-Kemmer matrices. As in the case of Dirac particles (see [7,10]), this can be done in various ways. In the present case it is convenient to use the expressions obtained above for the wave functions written in matrix form. Taking the wave functions of the initial state

$$\psi^{(\varepsilon_1)}(p_1) = \frac{1}{\varkappa\sqrt{2}}(\hat{p}_1 - i\varkappa)e_\mu^{(\varepsilon_1)}e^{\mu 1}, \qquad \varepsilon_1 = \pm, \tag{22}$$

and of the final state

$$\psi^{(\varepsilon_2)}(p_2) = \frac{1}{\varkappa\sqrt{2}}(\hat{p}_2 - i\varkappa)e_\nu^{(\varepsilon_2)}e^{\nu 1}, \qquad \varepsilon_2 = \pm, \tag{23}$$

and carrying out in reverse order all the operations used above in obtaining these expressions for the wave functions from the projection matrix dyads $\Lambda_{p_1}^{(\varepsilon_1)}$ and $\Lambda_{p_2}^{(\varepsilon_2)}$, we find for the desired matrix dyad entering into (21):

$$\psi^{(\varepsilon_1)}(p_1)\cdot\psi^{(\varepsilon_2)}(p_2) = -\frac{1}{2\varkappa^2}(\hat{p}_1 - i\varkappa)e_\mu^{(\varepsilon_1)}e_\nu^{(\varepsilon_2)}e^{\mu\nu}(\hat{p}_2 - i\varkappa). \tag{24}$$

Thus, the general expression for the matrix element M (21), after substitution of (24) with allowance for relation [11]

$$e^{\mu\nu} = (\delta_{\mu\nu} - \beta_\nu\beta_\mu)(3-\beta^2) = (\delta_{\mu\nu} - \beta_\nu\beta_\mu)P, \tag{25}$$

takes the following covariant form:

$$\mathrm{Sp}\left\{Q\psi^{(\varepsilon_1)}(p_1)\cdot\psi^{(\varepsilon_2)}(p_2)\right\} =$$

$$= \frac{1}{2\varkappa^2}\,\mathrm{Sp}\left\{Q(\hat{p}_1 - i\varkappa)\left(\hat{e}^{(-\varepsilon_2)}\hat{e}^{(\varepsilon_1)} - e^{(\varepsilon_1)}e^{(-\varepsilon_2)}\right)P(\hat{p}_2 - i\varkappa)\right\}, \tag{26}$$

where $P = 3-\beta^2 = 3-\beta_\rho\beta_\rho$ is the projection operator selecting the vector part of the 10-dimensional space [11].

For illustration, we give the calculation of the matrix element for the process of scattering of a vector particle by a fixed force center, with a vector character of the interaction ($Q = \beta_\mu$). According to (26), in this case we obtain (retaining under the sign Sp only terms containing products of an even number of matrices β_μ)

$$\begin{aligned} M^{\varepsilon_1\to\varepsilon_2}_{p_1\to p_2} &= \frac{i}{2\chi}\,\mathrm{Sp}\{(\beta_\mu\hat{p}_1P + \beta_\mu\hat{p}_2\bar{P})\,e^{(\varepsilon_1)}e^{(-\varepsilon_2)} \\ &\qquad - (\beta_\mu\hat{p}_1e^{(-\varepsilon_2)}e^{(\varepsilon_1)}P + \beta_\mu e^{(-\varepsilon_2)}e^{(\varepsilon_1)}p_2\bar{P})\} \\ &= \frac{i}{2\chi}\left[e^{(\varepsilon_1)}e^{(\varepsilon_2)}(p_{1\mu} + p_{2\mu}) - p_1e^{(\varepsilon_2)}\cdot e_\mu^{(\varepsilon_1)} - pe^{(\varepsilon_1)}\cdot e_\mu^{(\varepsilon_2)}\right]. \end{aligned} \tag{27}$$

Here the relation $P\beta_\mu + \beta_\mu\bar{P} = 0$ has been taken into account, where $\bar{P} = \beta^2 - 2$ is the projection matrix selecting the tensor part of the 10-dimensional space, and also the general formulas for traces of products of the matrices β_μ, P, and $\bar{P}$ have been used [12].

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Institute of Physics
Academy of Sciences of the BSSR

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Note: Figure translations are in progress. See original paper for figures.