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Abstract

Full Text

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ON METHODS OF REFINEMENT BY HIGHER-ORDER DIFFERENCES AND h^2 -EXTRAPOLATION

(Presented by Academician A. A. Dorodnitsyn, 14 XII 1962)

1. The method of successive refinements by higher-order differences of a numerical solution of partial differential equations was proposed empirically by L. Fox⁽¹⁾. A justification of the refinement method for solving the Dirichlet problem for the Poisson equation in a domain with a smooth boundary was given in our papers^(2,3). The refinement method, in contrast to ordinary mesh methods, makes it possible to obtain an approximate solution with an error whose order of smallness with respect to the mesh step h is the higher, the greater the number of bounded derivatives possessed by the desired solution. In the present note we present a method of refinement by higher-order differences for the solution of a mixed boundary-value problem for the Poisson equation in a rectangle, and state a theorem giving a justification of this method. In particular, for the approximate solution of problem (4)–(5) we have obtained the following representation on a square mesh:

$$v^q = v + O(h^{2q}) + O(h^{2m+1-q}), \quad (1)$$

where v^q is the approximate solution after the q -th refinement, and v is the solution of problem (4)–(5), possessing bounded derivatives of order $2m + 2$. According to (1), the error having the greatest order of smallness with respect to h is obtained for $q = [(2m + 1)/3]$ or $q = [(2m + 4)/3]$ (here $[]$ denotes the integer-part sign).

When solving boundary-value problems in a rectangle, on the one hand, the presence of corner points introduces a certain specificity into the construction of solution methods that take into account the character of the behavior of the solution at the corner points; but, on the other hand, in contrast to the case of a curvilinear boundary, there are no nonstandard mesh nodes situated near the boundary and, consequently, no complicated difference operators at boundary-adjacent mesh nodes, thanks to which the refinement method for a rectangular domain is more convenient in practice.

We point to the paper⁽⁴⁾, in which a refinement method is considered with the use of extrapolation of the desired solution beyond the limits of the given domain, and error estimates for the method are derived essentially only for

solving the Dirichlet problem in a square with zero boundary conditions, under the assumption that the right-hand side of the equation (and, consequently, the solution) is extended periodically from the square in an odd manner as a sufficiently smooth function on the whole plane.

In the present note, using the refinement method, an approximate solution is constructed for the Dirichlet problem and a mixed boundary-value problem for the Poisson equation in a rectangle, with nonhomogeneous boundary conditions and an arbitrary sufficiently smooth right-hand side. In this case the smoothness requirements in the initial problem (1)–(2) are imposed not on the desired solution u , but on the known functions, i.e., on the right-hand side of the equation and the boundary functions. A method is indicated for decomposing the desired solution into two components. The first component $F + P + Q$, which may contain singularities of the desired solution at the corner points, is constructed in

in explicit form by means of a finite combination of elementary functions. The second component v is smooth and can be approximately found by the method of refinements. Here a variant is proposed of the method of refinements by differences of higher orders, not requiring extrapolation of the solution beyond the given domain and using difference operators of finite order. The method of refinements is extended to the case of a triangular domain with possible angles $\pi/2$, $\pi/3$, $\pi/4$, and $\pi/6$. In addition, the error estimates ^(2, 3) of the method of refinements are improved for the case of a domain with a smooth boundary.

The expansion of the error in powers of h obtained below (Theorem 1, $m \geq 2$, $q = 1$) also makes it possible to apply, for refining an approximate solution constructed for different h , the method of h^2 -extrapolation and its generalization (see ⁽⁵⁾).

2. Let in the rectangle $R\{0 < x < a; 0 < y < b\}$ the equation be given

$$\Delta u = f(x, y) \tag{1}$$

with boundary conditions on the boundary Γ

$$\nu_j u|_{\Gamma_j} + \bar{\nu}_j u'_n|_{\Gamma_j} = \nu_j \varphi_j + \bar{\nu}_j \psi_j, \quad j = 1, 2, 3, 4, \tag{2}$$

where Γ_j are the sides of the rectangle (including the corner points); u'_n is the derivative along the inner normal; φ_j, ψ_j are functions continuous on Γ_j ; ν_j can take the values 0 or 1; $\bar{\nu}_j = 1 - \nu_j$. It is assumed that

$$\sum_{j=1}^4 \nu_j > 0.$$

Lemma 1. If $f \in C_{2m,\lambda}(\bar{R})^*$, $\bar{v}, \psi_j \in C_{2m+2,\lambda}(\Gamma_j)$, $\bar{v}_j \psi_j \in C_{2m+1,\lambda}(\Gamma_j)$, $m \geq 0$, $0 < \lambda < 1$, then the solution of problem (1)–(2) can be represented in the form

$$u = F + P + Q + v, \quad (3)$$

where F is a function which is a linear combination of the real and imaginary parts of the functions $(z - z_j)^\mu \ln(z - z_j)$ ($\mu = 0, 1, \dots, 2m + 2$; $j = 1, 2, 3, 4$; $z = x + iy$; $z_j = x_j + iy_j$ are the complex coordinates of the vertices of \bar{R}); P is a polynomial in x and y of degree $8m + 4$; Q is a harmonic polynomial of degree $8m + 11$; $v \in C_{2m+2,\lambda}(\bar{R})$ and satisfies the equation

$$\Delta v = g \quad (g = f - \Delta P) \quad (4)$$

under the boundary conditions

$$\nu_j v|_{\Gamma_j} + \bar{\nu}_j v'_n|_{\Gamma_j} = \nu_j \xi_j + \bar{\nu}_j \eta_j, \quad j = 1, 2, 3, 4, \quad (5)$$

$$\left(\xi_j = \varphi_j - (F + P + Q)|_{\Gamma_j}; \eta_j = \psi_j - (F + P + Q)'_n|_{\Gamma_j} \right),$$

and all derivatives up to order $2m$ inclusive of g , up to order $(2m + 1)$ of $\bar{\nu}_j \eta_j$, and up to order $(2m + 2)$ of $\nu_j \xi_j$, at the corner points of \bar{R} , vanish.

The function $F = \sum_{j=1}^4 F_j$, having at the corner points derivatives up to order $(2m + 2)$ inclusive with the same singularities as u , is constructed explicitly in ⁽⁶⁾. The polynomial P is constructed by means of the Hermite interpolation formula so that the derivatives of g vanish at the corner points. The harmonic polynomial Q is chosen by the method described in ⁽⁷⁾ so that the derivatives of the boundary functions vanish at the corner points. The regularity of v in \bar{R} is established by Theorem 1 of ⁽⁶⁾.

* $f \in C_{k,\lambda}(E)$, if f is continuously differentiable k times in E

and all its k -th derivatives satisfy in E a Hölder condition with exponent λ .

- Suppose that a/b is rational, and construct in \bar{R} a square grid with step h . Consider on the grid the following sequence of systems of difference equations

$$\begin{aligned} \Delta_h \vartheta^q &= -\frac{h^2}{4} g + D_{2m+1} \vartheta^{q-1} \quad \text{in } R, \\ \bar{\Delta}_h \vartheta^q &= \nu_j \xi_j - \bar{\nu}_j \left(\frac{h}{2} \eta_j + \frac{h^2}{4} g - \bar{D}_{2m+1} \vartheta^{q-1} \right) \quad \text{on } \Gamma - \bigcup_{k \neq j} \Gamma_k; \end{aligned} \quad (6)$$

$$\dot{\Delta}_h \vartheta^q = \nu_j \xi_j + \bar{\nu}_j \nu_{j+1} \xi_{j+1} - \bar{\nu}_j \bar{\nu}_{j+1} \left[\frac{h}{2} (\eta_j + \eta_{j+1}) + \frac{h^2}{4} g - \dot{D}_{2m+1} \vartheta^{q-1} \right] \quad \text{on } \Gamma_j \cap \Gamma_{j+1}.$$

Here $j = 1, 2, 3, 4$; $q = 1, 2, \dots, m$; $\vartheta^0 \equiv 0$; $\Delta_h \vartheta^q \equiv \vartheta^q(x, y) - [\vartheta^q(x+h, y) + \vartheta^q(x-h, y) + \vartheta^q(x, y+h) + \vartheta^q(x, y-h)]/4$; $\bar{\Delta}_h \vartheta^q \equiv \vartheta^q - \bar{\nu}_j [\vartheta_n^q + (\vartheta_s^q + \vartheta_{-s}^q)/2]/2$; $\dot{\Delta}_h \vartheta^q \equiv \vartheta^q - \bar{\nu}_j \bar{\nu}_{j+1} (\vartheta_s^q + \vartheta_{-s}^q)/2$; ϑ_n^q is the value of ϑ^q at the neighboring grid point on the normal; ϑ_s^q and ϑ_{-s}^q are the values of ϑ^q at neighboring points on Γ ; $h \leq \min\{a, b\}/(2m+1)$; D_{2m+1} , \bar{D}_{2m+1} , and \dot{D}_{2m+1} are linear difference operators of order not higher than $2m+1$, which, when substituted into equations (6) in place of ϑ^q and ϑ^{q-1} , give for the function v a residual of order $O(h^{2m+2})$. The operator \dot{D}_{2m+1} , at all nodes where possible, is expressed in terms of central differences, while at the remaining nodes the operator D_{2m+1} , as well as the operators \bar{D}_{2m+1} and \dot{D}_{2m+1} , are constructed by means of Newton's interpolation formula, using only nodes lying in \bar{R} (3).

Theorem 1. The solutions ϑ^q of the systems of difference equations (6) ($m \geq 1$; $q = 1, 2, \dots, m$) can be represented in the form

$$\vartheta^q = v + \sum_{k=q}^m h^{2k} w_k^q + O(h^{2m+1-q}), \quad (7)$$

where v is the solution of problem (4)–(5); w_k^q are the values on the grid of certain functions $w_k^q(x, y) \in C_{2m+2-2k, \lambda}(\bar{R})$, independent of h .

The proof is carried out with essential use of the properties, established by Lemma 1, of the functions g , $\nu_j \xi_j$, and $\nu_j \eta_j^*$ and is based on Theorem 1 of paper (6) and Lemma 2.

Lemma 2. The solution of the system of difference equations

$$\begin{aligned} \Delta_h \omega &= D_{2m+1} \mu \quad \text{in } R; \\ \bar{\Delta}_h \omega &= \bar{\nu}_j \bar{D}_{2m+1} \mu \quad \text{on } \Gamma - \bigcup_j (\Gamma_j \cap \Gamma_{j+1}); \end{aligned} \quad (8)$$

$$\dot{\Delta}_h \omega = \bar{\nu}_j \bar{\nu}_{j+1} \dot{D}_{2m+1} \mu \quad \text{on } \Gamma_j \cap \Gamma_{j+1}$$

for $|\mu(x, y)| < h^\beta$ satisfies in \bar{R} the inequality

$$|\omega| < c_0 h^{\beta-1}, \quad (9)$$

where $c_0 = c_0(a, b, m)$ is a constant independent of h .

According to Theorem 1, the error of the approximate solution ϑ^q of problem (4)–(5) decomposes into two components: a regular component, represented as a polynomial in h^2 with coefficients that are values of smooth functions, with leading term of order h^{2q} , and an irregular component of order h^{2m+1-q} .

4. The method of refinement by differences of higher orders can be applied directly for the approximate computation of the function $u - F$ without pre-

* When solving the Dirichlet problem in a rectangle, it suffices that, at the corner points, the derivatives of g and ξ_j vanish up to the $2m$ -th and $(2m+2)$ -th orders, respectively, only with an even number of differentiations with respect to x and with respect to y .

preliminary separation of the polynomials P and Q . In doing so, in the system of difference equations for the q -th approximation it will be necessary to vary the boundary conditions by specially chosen functions of order h^{2q} .

5. The method of refinements can also be carried out when using difference schemes of other types. For example, when using the multipoint difference schemes proposed in ^(8,6), the regular component of the error in the q -th approximation will have order $O(h^{6q})$, while the irregular component will have order $O(h^{2m+1-q})$.
6. Using the remarkable properties of the solution of the Dirichlet problem for the Laplace equation in a domain with angles π/j ($j = 2, 3, \dots$) (see ⁽⁹⁾), the method of refinements can be applied on a square, rectangular, and triangular grid in a triangle with possible angles $\pi/2$, $\pi/3$, $\pi/4$, and $\pi/6$.
7. Lemma 2 extends to the systems of difference equations considered in ^(2,3) in solving the Dirichlet problem in a domain with a smooth boundary, and improves the estimates of the irregular error, which in the q -th approximation will have order $O(h^{2n-1-q})$, and not $O(h^{2n-2q})$, as formulated in Theorems II of ^(2,3).

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