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Abstract

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STATIONARY AXISYMMETRIC MODEL OF A CUMULUS CLOUD

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Generalizing the work ([1,2]), we indicate a way of solving the stationary axisymmetric nonlinear problem of a cumulus cloud. Under certain assumptions ([2]) one may take as the initial system:

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = -R\theta_0\frac{\partial}{\partial r}\left(\frac{p'}{P}\right) + \nu_z\frac{\partial^2 u}{\partial z^2} + \nu\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial ur}{\partial r};$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -R\theta_0\frac{\partial}{\partial z}\left(\frac{p'}{P}\right) + \lambda\vartheta + \nu_z\frac{\partial^2 w}{\partial z^2} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right);$$

$$u\frac{\partial \vartheta}{\partial r} + w\frac{\partial \vartheta}{\partial z} = \alpha(z)w + k_z\frac{\partial^2 \vartheta}{\partial z^2} + \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \vartheta}{\partial r}\right) \quad \left(\lambda = \frac{g}{\theta_0}\right); \tag{1}$$

$$\frac{\partial ur}{\partial r} + \frac{\partial wr}{\partial z} = 0; \qquad \alpha(z) = \begin{cases} \gamma - \gamma_{\text{B}}, & 0 \le z \le h, \\ \gamma - \gamma_{\text{a}}, & z > h \end{cases} \quad \left(\gamma = -\frac{\partial \theta}{\partial z} = \text{const}\right),$$

where r, z are the radial and vertical coordinates (z is directed upward); u, w are the radial and vertical components of velocity; ϑ, p' are deviations of temperature and pressure from $\theta(z), P(z)$, the values of these meteorological elements at a sufficient distance from the cloud, where air motion is absent; ν_z, ν are the vertical and horizontal kinematic coefficients of turbulent viscosity; k_z, k are the vertical and horizontal coefficients of turbulent thermal conductivity (we assume that $\nu_z = \nu = k_z = k = \text{const}$); g is the acceleration of gravity, $\theta_0 = \text{const}$ is the mean value of the absolute air temperature; R is the gas constant for air; $\gamma_{\text{a}}, \gamma_{\text{B}}$ are the dry- and moist-adiabatic lapse rates; h is the thickness of the layer of moist instability, which we regard as known and independent of r (we place the origin of coordinates at the lower boundary of this layer). In what follows, with a view to obtaining an analytical solution, we shall put $\gamma - \gamma_{\text{B}} = \gamma_{\text{a}} - \gamma = a_0 = \text{const}$. The error introduced thereby lies within the accuracy of the problem.

For further work it is expedient to pass to dimensionless quantities (letters with bars) from the relations

$$r = \frac{\sqrt{\nu}}{\sqrt[4]{a_0\lambda}}\,\bar{r}; \qquad w = h\sqrt{a_0\lambda}\,\bar{w}; \qquad u = \sqrt{a_0\lambda\nu^2}\,\bar{u}; \qquad p' = \frac{P\nu\sqrt{a_0\lambda}}{R\theta}\,\bar{p}. \quad (2)$$

$$z = h\bar{z}; \qquad \vartheta = a_0 h\bar{\vartheta}; \qquad \alpha = a_0\bar{\alpha}(\bar{z}) = a_0 \begin{cases} 1, & 0 \le z \le 1, \\ 0, & z > 1. \end{cases}$$

Then system (1) takes the form (we shall not write bars over the letters):

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial r} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial ur}{\partial r}\right) + \delta\frac{\partial^2 u}{\partial z^2}; \quad (3)$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\delta\frac{\partial p}{\partial z} + \vartheta + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \delta\frac{\partial^2 w}{\partial z^2},$$

$$u\frac{\partial \vartheta}{\partial r} + w\frac{\partial \vartheta}{\partial z} = \alpha(z)w + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \vartheta}{\partial z}\right) + \delta\frac{\partial^2 \vartheta}{\partial z^2}. \quad (4)$$

We do not write the continuity equation, since it retains its form. The dimensionless parameter $\delta = \nu/\sqrt{a_0\lambda h^2} \ll 1$. Indeed, if

put $a_0 = 3{\cdot}10^{-3}$ deg$\cdot$m^{-1}, $\lambda = 3{\cdot}10^{-2}$ msec^{-2}deg^{-1}, $h = 10^3$ m; $\nu = 10^3$ m^{2}sec^{-1} (according to (3), in real clouds $\nu = 300$ m^{2}sec^{-1}), then we obtain $\delta = 0.1$. This circumstance permits us, without introducing any substantial error in solving the problem, to discard in (3), (4) the terms having the factor δ (which we shall do), thereby carrying out a simplification of the theory of the boundary layer (directed along the cloud axis).

As boundary conditions for the simplified system we shall take:

$$u\big|_{r=0} = \frac{\partial \vartheta}{\partial r}\bigg|_{r=0} = \frac{\partial w}{\partial r}\bigg|_{r=0} = w\big|_{r=\infty} = v\big|_{r=\infty} = p\big|_{r=\infty} = 0; \qquad \int_0^\infty wr\,dr < \infty; \quad (5)$$

$$\vartheta = w = 0 \quad \text{for } z = 0. \quad (6)$$

The meaning of conditions (5) is obvious. Condition (6) follows from (2). The problem is now posed. The course of its solution is clear: first we find w and ϑ from (4), then u from the equation of continuity and, finally, p from (3). As was noted already in (1^2), the problem thus posed is homogeneous and therefore has the trivial zero solution, corresponding to complete rest.

Let us now consider a possible way of finding a nontrivial solution, which we shall interpret as a theoretical model of a cumulus cloud. First of all note that in what follows it is convenient to operate with the new variable $\xi = 0.25r^2$ and with the stream function ψ, related to the velocity by the well-known relations

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial z} = -\frac{2}{\sqrt{\xi}}\frac{\partial\psi}{\partial z}; \qquad w = \frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{1}{2}\frac{\partial\psi}{\partial\xi}. \tag{7}$$

We begin with the region $0 \le z \le 1$, where we shall seek the solution in the form

$$\psi = 2zF(\xi) \qquad (0 \le z \le 1). \tag{8}$$

Then equations (4) lead to the following ordinary equation:

$$(\xi F'')' + FF'' - {F'}^2 + F' = 0, \tag{9}$$

whose solution, according to (5), (6), must satisfy the conditions

$$F = \sqrt{\xi}\,F'' = 0 \quad \text{as } \xi \to 0; \qquad F' < \infty, \quad F < \infty \quad \text{as } \xi \to \infty. \tag{10}$$

The solution of problem (9), (10), as well as a table of values of the function $F(\xi)$, are given in (4)*. Having $F(\xi)$, by means of (7) we find: $w = \vartheta = zF'(\xi)$ $(0 \le z \le 1)$. In the region $z > 1$ the equations apparently will have to be solved numerically. In doing so, in addition to (5), one must require the fulfillment of the conditions:

$$z = 1, \quad \vartheta = w = F'(\xi). \tag{11}$$

The following path of solution seems possible to us. Passing in (4) to the new independent variables ψ, z, we obtain the equations

$$\frac{\partial\vartheta}{\partial z} = -1 + \frac{\partial}{\partial\psi}\xi w\frac{\partial\vartheta}{\partial\psi}; \qquad \frac{\partial w}{\partial z} = \frac{\vartheta}{w} + \frac{\partial}{\partial\psi}\xi w\frac{\partial w}{\partial\psi} \quad \left(\xi = \int_0^{\psi}\frac{d\psi}{w}\right), \tag{12}$$

which, as is easy to see, are very similar to diffusion equations. Therefore one may hope that the application to (12) of one of the numerical methods developed for parabolic-type equations will be successful. As for the boundary conditions for (12), they can be obtained from (5), (11). We have:

$$\text{for } \psi = 0 \quad \partial w/\partial\psi < \infty, \quad \partial\vartheta/\partial\psi < \infty; \qquad \text{for } \psi = \psi_\infty(z) \quad w = \vartheta = 0, \tag{13}$$

$$\text{for } z = 1 \quad w = \vartheta = W(\psi). \tag{14}$$

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* In the present article the variable ξ differs from ζ in (4) by the presence of the factor $a = 1.144$.

Here $\psi_\infty(z)$ is the value of ψ as $\xi \to \infty$. This function must be found together with w and ϑ so that the relations

$$\frac{d}{dz}\int_0^{\psi_\infty(z)} \vartheta\, d\psi = -\psi_\infty(z); \qquad \frac{d}{dz}\int_0^{\psi_\infty(z)} w\, d\psi = \int_0^{\psi_\infty(z)} \frac{\vartheta}{w}\, d\psi, \tag{15}$$

hold, to which (12) lead after their integration with respect to ψ from 0 to ψ_∞. The function $W(\psi)$ is obtained by eliminating ξ from conditions (11). The way to determine the quantities u and p is obvious.

In the present note we do not present the results of numerical integration and shall confine ourselves to obtaining, in analytical form, the solution of a simpler problem, namely one in which the turbulent terms are not taken into account in equations (4). Again let us begin with the region $0 \le z \le 1$. As before, we shall seek the solution in the form (8). However, now instead of (9) we have $FF'' - {F'}^2 + F' = 0$. The solution of this equation satisfying conditions (10) will be $F = n[1 - \exp(-\xi/n)]$, where n is an arbitrary constant. Recalling that the solution of equation (10) gave $F(\infty) = 5.19$ (see (4)), we set $n = 5.19$, thereby roughly taking into account the influence of turbulence.

Indeed, comparing the curve $F'(\xi)$ obtained from the exact equation (10) (see (4)) and the approximate curve $F'(\xi) = \exp(-\xi/n)$, we are convinced that the effect of turbulence on the macromotions of the air for $z \le 1$ is essentially reduced to fixing certain horizontal dimensions of the cloud. (In addition, turbulence somewhat smooths, near the axis of symmetry, the horizontal profiles of vertical velocity and temperature.)

For simplicity of notation we put

$$\zeta = \xi/n; \qquad \varphi = \psi/2n = z(1 - e^{-\zeta}). \tag{16}$$

Then the solution for $0 \le z \le 1$ takes the form

$$w = \vartheta = \partial\varphi/\partial\zeta = ze^{-\zeta}; \qquad u = -\sqrt{n/\zeta}\,\partial\varphi/\partial z = -\sqrt{n/\zeta}\,(1 - e^{-\zeta}). \tag{17}$$

In the region $z > 1$, according to (12)–(14), one must solve the equations $\partial\vartheta/\partial z = -1$; $\partial w/\partial z = -\vartheta/w$, satisfying the conditions $w = \theta = 1 - \varphi$ at $z = 1$. We easily find

$$\vartheta = 2 - z - \varphi; \qquad w = \sqrt{(z-\varphi)^2 - 2(z-1)^2}, \quad z > 1. \tag{18}$$

Expression (18), obviously, has meaning only where w is real. Therefore, equating the radicand in (18) to zero, we obtain $\varphi_\infty(z) = \sqrt{2} - (\sqrt{2} - 1)z$, where φ_∞ is the value of φ at the boundary of the cloud. Using (18), by quadrature we find

$$\zeta = \int_0^\varphi \frac{d\varphi}{w} = \ln\frac{z + w_0}{z - \varphi + w}, \tag{19}$$

where $w_0 = w_{\varphi=0} = \sqrt{4z - z^2 - 2}$ describes the distribution of vertical velocity on the axis of symmetry of the cloud for $z > 1$. Solving (19) with respect to φ, we obtain the explicit expression for the stream function

$$\varphi = w_0 \operatorname{sh}\zeta - z(\operatorname{ch}\zeta - 1), \tag{20}$$

which has meaning as long as $\zeta < \zeta_\infty = \ln(z + w_0) - \ln\sqrt{2}(z - 1)$. (We arrive at the formula for ζ_∞ by substituting in (19) $\varphi = \varphi_\infty,\ w = 0$.) From (20), (17) we find explicit expressions for w and u:

$$w = \partial\varphi/\partial\zeta = w_0 \operatorname{ch}\zeta - z \operatorname{sh}\zeta; \tag{21}$$

$$u = -\sqrt{\frac{n}{\zeta}}\,\frac{\partial\varphi}{\partial z} = \sqrt{\frac{n}{\zeta}}\left(\operatorname{ch}\zeta + \frac{z-2}{w_0}\operatorname{sh}\zeta - 1\right). \tag{22}$$

Substituting $\zeta = \zeta_\infty$ in (22), we find the value of u at the cloud boundary

$$u_{\zeta=\zeta_\infty} = (\sqrt{2} - 1)\sqrt{n/\zeta_\infty} \qquad (z > 1).$$

Therefore it must be assumed that outside the cloud

$$u = (\sqrt{2} - 1)\sqrt{n/\zeta},$$

Fig. 1

Figure 1: Fig. 1

and, consequently, the streamlines there are horizontal. Of course, this expression satisfies both the continuity equation and equations (4).

The results of calculating the spatial distribution of w, ϑ, and ψ from formulas (16), (18), (20), (21) are given in Fig. 1, which is an axial section of the cloud model.

Fig. 1

Here the thin lines are streamlines (the arrows show the direction of air motion), the heavy lines are isolines of vertical velocity, and the dashed lines are isolines of temperature deviations. As the lower boundary of the cloud the isoline $w = 0.12$ is taken; at points I, II, III the quantities w and ϑ take extremal values: at point I, $\vartheta = \vartheta_{\text{max}} = 1.0$; at point II, $w = w_{\text{max}} = 1.41$ (at this same point, $\vartheta = 0$); at point III, $\vartheta = \vartheta_{\text{min}} = -1.41$. All quantities and coordinates in Fig. 1 are dimensionless. To pass to dimensional quantities one should use formulas (2). For the values of the parameters that were adopted in computing δ, we have: $w_{\text{max}} = 14\ \text{m}\,\text{sec}^{-1}$ at $z = 2$ km; $\vartheta_{\text{max}} = 3^\circ$ at $z = 1$ km; $\vartheta_{\text{min}} = -4.2^\circ$ at $z = 3.4$ km.

Having constructed the velocity and temperature field, it is not difficult, by analogy with the way this was done in ([2]), to obtain the spatial distribution of pressure and water content in the cloud, and also to calculate the maximum possible amount of precipitation.

In conclusion, we note that at present, under the direction of Prof. G. K. Sulakvelidze, the theory set forth is being compared with observational materials obtained in recent years at the High-Mountain Geophysical Institute of the Academy of Sciences of the USSR. The preliminary results have proved encouraging. In particular, this can be asserted with respect to the distribution of w with height ([5,6]).

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Note: Figure translations are in progress. See original paper for figures.