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Abstract

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ON CONDITIONS FOR THE REPRESENTABILITY OF A FUNCTION OF MANY VARIABLES IN THE FORM OF A SUM OF A FINITE NUMBER OF PLANE WAVES OF GIVEN DIRECTIONS

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Let in Euclidean space $R_n = \{\mathbf{x}\}$, $\mathbf{x} = (x_1, \dots, x_n)$, there be given a system of pairwise noncollinear vectors $\mathbf{a}_i = (a_{i1}, \dots, a_{in})$, $i = 1, \dots, k$, and some domain D.

Denote by $F_m[D]$ the totality of all functions having in the domain D continuous partial derivatives up to order m inclusive, and by $C_m[\mathbf{a}_i, D]$ the totality of all functions $\varphi_i(t_i)$, $i = 1, \dots, k$, that are m times continuously differentiable on the interval

$$\inf_{\mathbf{x}\in D}(\mathbf{a}_i\mathbf{x}) < t_i < \sup_{\mathbf{x}\in D}(\mathbf{a}_i\mathbf{x}), \qquad \mathbf{a}_i\mathbf{x} = a_{i1}x_1 + \dots + a_{in}x_n, \tag{1}$$

In the present paper we derive necessary, and under certain restrictions imposed on the domain D, sufficient conditions for there to exist, for a function $f(\mathbf{x})$, $f(\mathbf{x}) \in F_m[D]$, where m is a sufficiently large natural number, functions $\varphi_i(t_i) \in C_m[\mathbf{a}_i, D]$, $i = 1, \dots, k$, such that in the domain D

$$f(\mathbf{x}) = \sum_{i=1}^{k} \varphi_i(\mathbf{a}_i\mathbf{x}). \tag{2}$$

The indicated necessary and sufficient conditions are obtained in the form of a system of partial differential equations with one unknown function. In what follows in the paper, starting from the set $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$, a special system of differential operators is constructed. The constructed operators are applied to a function $f(\mathbf{x})$ representable in the form (2). Specifying part of these operators on a segment $[\mathbf{x}_0, \mathbf{x}_1]$ of a straight line not orthogonal to any of the directions $\mathbf{a}_1, \dots, \mathbf{a}_k$, and the remaining operators at some point of this segment, determines the function $f(\mathbf{x})$ uniquely. The domain of its definition is the intersection of the strips

$$\mathbf{a}_i\mathbf{x}_0 \leq \mathbf{a}_i\mathbf{x} \leq \mathbf{a}_i\mathbf{x}_1, \qquad i = 1, \dots, k, \qquad \mathbf{x} \in R_n. \tag{3}$$

Let us first note that the coordinates of the vectors $\mathbf{a}_i = (a_{i1}, \dots, a_{in})$, $i = 1, \dots, k$, may be regarded as homogeneous coordinates of points of the $(n-1)$-dimensional projective space Π_{n-1}. Denote by $\mathbf{a}$ an arbitrary point of this space, determined by homogeneous coordinates $a_1, \dots, a_n$. Let some set of points M be given in the space Π_{n-1}. Consider, in the ring of polynomials $K[\mathbf{y}] = K[y_1, \dots, y_n]$ over the field of real numbers, the totality $\{P(\mathbf{y})\}$ of all forms (homogeneous polynomials) $P(\mathbf{y})$, each of which contains the set M (vanishes at every point of the set M). Among these forms (see (1)) there exists a finite number of forms

$$P_{iM}(\mathbf{y}), \qquad i = 1, \dots, s, \tag{4}$$

such that for every homogeneous polynomial $P(\mathbf{y})$ containing the set M, the equality

$$P(\mathbf{y}) = \sum_{i=1}^{s} R_i(\mathbf{y}) P_{iM}(\mathbf{y}), \tag{5}$$

will hold,

where $R_i(y)$ are certain forms from the ring $K[y]$. In what follows we assume that the system (4) is minimal: none of the polynomials of this system is expressed in terms of the others by formula (5). Any such system of polynomials will be called a basis belonging to the set M. Replacing in each term of the polynomial $P(y)$ the power $y_p^{\nu_p}$ by the symbolic power $\partial^{\nu_p}/\partial x_p^{\nu_p}$, $p = 1, \dots, n$, we obtain the differential operator $P(\partial/\partial x)$ corresponding to the form $P(y)$. For this operator the representation

$$P\left(\frac{\partial}{\partial x}\right) = \sum_{i=1}^{s} R_i\left(\frac{\partial}{\partial x}\right) P_{iM}\left(\frac{\partial}{\partial x}\right). \tag{5'}$$

is valid.

Using the basis (4), we construct a system of differential equations with one unknown function

$$P_{i,M}(\partial/\partial x)u = P_{iM}(\partial/\partial x_1, \dots, \partial/\partial x_n)u = 0, \quad i = 1, \dots, s. \tag{6}$$

Lemma 1. *If the set $\{P(y)\}$ consists only of the identically zero form, then for any homogeneous polynomial $Q(x)$ of arbitrary degree m there exist real numbers λ_i and, in the set M, points a_i, $i = 1, \dots, r$, such that for all $x \in R_n$*

$$Q(x) = \sum_{i=1}^{r} \lambda_i (a_i x)^m. \tag{7}$$

If, however, the set $\{P(y)\}$ contains at least one nontrivial form, then in order that a homogeneous polynomial $Q(x)$ be representable in the form (7), it is necessary and sufficient that this polynomial satisfy the system (6).

The validity of the first assertion of the lemma was proved in the note (^{2}). The necessity of the second is obvious. Its sufficiency is proved analogously to how this was done in the paper (^{3}) for the special case $Q(x) = (ax)^m$.

Suppose now that M is a finite set of points of P_{n-1}. The **index of a point** a with respect to the set M will mean the least of the degrees of forms of the ring $K[y]$ which contain the set M but do not contain the point a. A form containing the set M and not containing the point a, whose degree is equal to the index of the point a with respect to the set M, will be called the **index form** for the point a and the set M and will be denoted by $\Gamma_{aM}(y)$. The corresponding differential operator $\Gamma_{aM}(\partial/\partial x)$ will be called the **index operator** for the indicated point and set.

Theorem 1. *Let $M = \{a_1, \dots, a_k\}$. If every section of the domain D by a hyperplane orthogonal to the vector a_i is connected, $i = 1, \dots, k$, then for the given set M there exists a natural number $l = l(M)$ such that every function $f(x)$, $f(x) \in F_m[D]$, $m \geq l$, satisfying in the domain D the system (6), is represented in this domain in the form (2), moreover $\varphi_i(t_i) \in C_m[a_i, D]$.*

Conversely, if for a function $f(x)$ in some domain the equality (2) is valid, where $\varphi_i(t_i) \in C_m[a_i, D]$, $m \geq l$, then in this domain $f(x)$ satisfies the system (6).

We shall prove the direct assertion by induction on the number of points k. For $k = 1$, $M = \{a_1\}$, we put $P_{i\{a_1\}}(y) = e_i y$, $i = 1, \dots, n-1$, where $\{e_1, \dots, e_{n-1}\}$ is an arbitrary collection of linearly independent vectors orthogonal to the vector a_1. The system (6) is written in the form

$$P_{i\{a_1\}}(\partial/\partial x)u = (e_i, \partial/\partial x)u = 0, \quad i = 1, \dots, n-1,$$

and in this case the theorem is easily proved. Assuming it true for a set consisting of $k-1$ points, we construct, for the set $M_1 = \{a_2, \dots, a_k\}$, the forms $P_{iM_1}(y)$, $i = 1, \dots, r$, and consider the products $P_{iM_1}(y) \cdot P_{j\{a_1\}}(y)$, $i = 1, \dots, r$; $j = 1, \dots, n-1$ (we assume here that the forms

$P_{j\{a_1\}}(\mathbf{y})$, $j = 1, \dots, n-1$, do not contain the points M_1. Each such product contains the set $M = \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$, and hence is expressed in terms of the polynomials $P_{\sigma M}(\mathbf{y})$, $\sigma = 1, \dots, s$, by formula (5). The latter means that any product

$$P_{iM_1}(\partial/\partial \mathbf{x}) \cdot P_{j\{a_1\}}(\partial/\partial \mathbf{x})$$

can be expressed in terms of the forms $P_{\sigma M}(\partial/\partial \mathbf{x})$, $\sigma = 1, \dots, s$, by means of equality ($5'$). Therefore every solution $u = f(\mathbf{x})$ of system (6), defined in the domain D, $f(\mathbf{x}) \in F_m[D]$, $m \geq l(M) \geq l(M_1) + 1$*, is in this domain a solution of the system

$$P_{iM_1}(\partial/\partial \mathbf{x}) \cdot P_{j\{a_1\}}(\partial/\partial \mathbf{x}) u = 0, \qquad i = 1, \dots, r;\ j = 1, \dots, n-1.$$

Hence, and from the induction assumptions, it follows that this solution $u = f(\mathbf{x})$ will satisfy in the domain D the system:

$$P_{j\{a_1\}}(\partial/\partial \mathbf{x}) u = \sum_{i=2}^{k} \varphi_{ij}(\mathbf{a}_i \mathbf{x}), \qquad j = 1, \dots, n-1, \tag{8}$$

where the functions $\varphi_{ij}(t_i) \in C_{m-1}[\mathbf{a}_i, D]$, $j = 1, \dots, n-1$; $i = 2, \dots, k$, and in the indicated domain are subject to the conditions

$$\sum_{i=2}^{k} \varphi'_{it}(\mathbf{a}_i \mathbf{x}) P_{q\{a_1\}}(\mathbf{a}_i) = \sum_{i=2}^{k} \varphi'_{iq}(\mathbf{a}_i \mathbf{x}) P_{t\{a_1\}}(\mathbf{a}_i), \qquad q, t = 1, \dots, n-1. \tag{9}$$

Using the relations (9), let us express the functions $\varphi_{ih}(t_i)$, $h = 1, \dots, n-2$, for example, in terms of the function $\varphi_{i\,n-1}(t_i)$, $i = 2, \dots, k$. To this end, putting $t = h$ and $q = n-1$, we act on both sides of (9) with the operator $\Gamma_{\alpha_i M - \{a_i\}}(\partial/\partial \mathbf{x})$. Then, integrating the result obtained $\alpha_i + 1$ times (α_i is the index of the point $\mathbf{a}_i$ with respect to the set $M - \{a_i\}$), we shall have

$$\varphi_{ih}(t_i) = \frac{P_{h\{a_1\}}(\mathbf{a}_i)}{P_{n-1\{a_1\}}(\mathbf{a}_i)} \varphi_{i\,n-1}(t_i) + Q_{ih}(t_i),$$

$$h = 1, \dots, n-2; \qquad i = 2, \dots, k,$$

where $Q_{ih}(t_i)$ is a certain polynomial in t_i of degree not exceeding α_i.

It can be shown that, when conditions (9) and (10) are fulfilled, every solution of system (8) of the class $F_m[D]$, $m \geq l$, has the form

$$u(\mathbf{x}) = \varphi_1(\mathbf{a}_1 \mathbf{x}) + \sum_{i=2}^{k} \frac{1}{P_{n-1\{a_1\}}(\mathbf{a}_i)} \int_{t_i^0}^{a_i \mathbf{x}} \varphi_{i\,n-1}(\tau)\, d\tau + T(\mathbf{x}),$$

where $\varphi_1(t_1) \in C_m[\mathbf{a}_1, D]$, $m \geq l$, $T(\mathbf{x})$ is a certain polynomial, and t_i^0 is a number from the interval (1). The function $f(\mathbf{x})$, being a solution of system (8), must be of the same form. It follows from this that the corresponding polynomial $T(\mathbf{x})$ is a solution of system (6). By virtue of Lemma 1 it must be representable by formula (2). The direct theorem is proved. The converse theorem is obvious.

Let $M \in \Pi_{n-1}$ be any set of points for which there exists a nontrivial form $P(\mathbf{y})$ containing it. Denote by $\gamma_1 < \dots < \gamma_\mu$ the sequence of all distinct degrees of the polynomials of some basis belonging to the set M, and by M_{γ_i} the algebraic variety consisting of the collection of all common zeros of those basis polynomials whose degrees are less than or equal to γ_i, $i = 0, 1, \dots, \mu$ $(\gamma_0 = 0,\ M_0 = \Pi_{n-1})$. The following lemmas are easily proved:

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* The number $l(M)$ is chosen, in addition, so that all derivatives of the functions under consideration which e

Lemma 2. Let $\gamma_i \leq m < \gamma_{i+1}$, $i = 0, 1, \dots, \mu$; $\gamma_{\mu+1} = \infty$. Then and only then does the degree $(\mathbf{ax})^m$ decompose according to formula (7), when $\mathbf{a} \in M_{\gamma_i}$.

We shall call the **index of a finite set of points** $M \in \Pi_{n-1}$ the greatest of the indices of the points of this set relative to their complements in the set M.

Lemma 3. Suppose that the set of points M has index γ, and let a point $\mathbf{a}$, $\mathbf{a} \in M$, have index γ relative to the set $M - \{\mathbf{a}\}$. Then the index of the set $M - \{\mathbf{a}\}$ is either γ or $\gamma - 1$. If some point $\mathbf{b} \in M$ has, relative to $M - \{\mathbf{b}\}$, index $\beta < \gamma$, then its index relative to the set $M - \{\mathbf{a}\} - \{\mathbf{b}\}$ is also equal to β.

Let again $M = \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$, and let $[\mathbf{x}_0, \mathbf{x}_1]$ be some segment of the line of the space R_n, not orthogonal to any of the directions $\mathbf{a}_i$, $i = 1, \dots, k$.

We now pose the problem of determining a sufficiently smooth function $f(\mathbf{x})$, representable in the form (2) in the domain which is the intersection of the strips (3), from the following data.

Take arbitrary continuous functions $\varphi_i(t_i)$, $\mathbf{a}_i\mathbf{x}_0 \leq t_i \leq \mathbf{a}_i\mathbf{x}_1$, $i = 1, \dots, k$. We first require that at each point $\mathbf{z}$ of the segment $[\mathbf{x}_0, \mathbf{x}_1]$ the equalities

$$\Gamma_{a_i M - \{\mathbf{a}_i\}}\left(\frac{\partial}{\partial \mathbf{x}}\right) f(\mathbf{x})\Big|_{x=\mathbf{z}} = \varphi_i(\mathbf{a}_i\mathbf{z}), \qquad i = 1, \dots, k, \tag{11}$$

hold, where $\Gamma_{a_i M - \{\mathbf{a}_i\}}(\partial/\partial\mathbf{x})$ is the index operator for the point $\mathbf{a}_i$ and the set $M - \{\mathbf{a}_i\}$.

Assume further that the index of the set M is equal to γ. In addition to the data (11), specify at some point of the segment $[\mathbf{x}_0, \mathbf{x}_1]$, for instance at the point $\mathbf{x}_0$, the following set of index operators from the function $f(\mathbf{x})$ of orders lower than γ.

Let M^1 be such a subset of M of index $\gamma - 1$ that adjoining to it any point of the set $M - M^1$ turns it into a set of index γ. In the set M^1 we single out all points $\mathbf{a}_i$ of index γ relative to the set $M - \{\mathbf{a}_i\}$. For each such point and its complement to M^1, consider the index operator from $f(\mathbf{x})$ and prescribe its value at the point $\mathbf{x}_0$. We obtain a system S_1 of homogeneous differential operators of order $\gamma - 1$ from the function $f(\mathbf{x})$, specified at the point $\mathbf{x}_0$.

By M^2 denote a subset of M^1 of index $\gamma - 2$ such that adjoining to it any point of the set $M^1 - M^2$ turns it into a set of index $\gamma - 1$. In the set M^2 we single out all points $\mathbf{a}_i$ of index $\gamma - 1$ relative to the set $M^1 - \{\mathbf{a}_i\}$. For these points and their complements to the set M^2 we consider the corresponding index operators from the function $f(\mathbf{x})$ and prescribe their values at the point $\mathbf{x}_0$. We obtain a system S_2 of homogeneous differential operators from the function $f(\mathbf{x})$ of order $\gamma - 2$, the values of which are specified at the point $\mathbf{x}_0$. Continuing in the same way, we specify systems of differential operators $S_3, \dots, S_\gamma$ of orders respectively $\gamma - 3, \dots, 1$ from the function $f(\mathbf{x})$ at the point $\mathbf{x}_0$. Finally, specify the value $f(\mathbf{x}_0)$. One can prove the theorem:

Theorem 2. The equalities (11) and the values of all operators from the set $S = S_1 + \dots + S_\gamma$ from the function $f(\mathbf{x})$ at the point $\mathbf{x}_0$, as well as the value $f(\mathbf{x}_0)$, together uniquely determine a sufficiently smooth function $f(\mathbf{x})$, representable by formula (2), in the domain which is the intersection of the strips (3).

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