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Abstract

Full Text

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ON ESTIMATES OF THE RATE OF CONVERGENCE OF SOLUTIONS OF DIFFERENCE EQUATIONS TO SOLUTIONS OF ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS AND ON ONE NUMERICAL METHOD FOR SOLVING THE DIRICHLET PROBLEM

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In the paper ([1]) (see also ([2])) it is shown that diffraction problems can be considered as problems of determining generalized solutions with finite energy integral for equations with discontinuous coefficients, and methods for their solution are given. From ([1],[2]), in particular, there follows the existence and uniqueness of the generalized solution of the following problem:

$$-\frac{\partial}{\partial x_1} a' \frac{\partial u^a}{\partial x_1} + \frac{\partial}{\partial x_2} a' \frac{\partial u^a}{\partial x_2} = f'(x_1, x_2),$$

$$a' = \begin{cases} 1, & x \in \Omega_1, \\ a, & x \in \Omega_2; \end{cases} \qquad f' = \begin{cases} f(x_1, x_2), & x \in \Omega_1, \\ 0, & x \in \Omega_2; \end{cases} \tag{1}$$

$$[u^a]_{S_1} = 0, \qquad \left[a' \frac{\partial u^a}{\partial n}\right]_{S_1} = 0, \qquad u^a\big|_{S_2} = 0.$$

Here a is a constant ($a \gg 1$); S_1 is a sufficiently smooth ([1],[2]) boundary of an arbitrary domain Ω_1; S_2 is the boundary of a rectangle Ω containing the domain Ω_1; Ω_2 is the domain enclosed between S_1 and S_2; n is the normal to S_1; [ ] denotes the jump of a function on the line.

For the solution of problem (1), as follows from ([1],[2]), the estimate

$$\|u^a\|^2_{W_2^1(\Omega_1)} + a\|u^a\|^2_{W_2^1(\Omega_2)} \leq C_1 \|f(x)\|^2_{L_2(\Omega_1)}, \tag{2}$$

holds, where C_1 depends only on the form of the domains Ω_1, Ω_2.

From ([2]) one easily derives the known fact that $u^a(x)$, as $a \to +\infty$, tends in the norm W_2^1 to zero in the domain Ω_2 and to the solution of the Dirichlet problem

$$\Delta u = f(x_1, x_2), \qquad u\big|_{S_1} = 0 \tag{3}$$

in the domain Ω_1.

On the other hand, for problem (1) it is shown in ([1],[2]) how to construct convergent finite-difference schemes. Since the domain Ω is a rectangle, there exist comparatively effective methods for solving the algebraic systems that arise in this connection. In view of this, for problem (3) in an arbitrary domain it was natural to analyze the following method of solution: enclose the domain Ω_1 in the rectangle Ω most closely adjacent to it, and in this rectangle solve problem (1) (taking $\Omega_2 = \Omega - \Omega_1$) by the finite-difference method with sufficiently small mesh size h and sufficiently large a.

From what has been said above it is clear that if one takes the approximate solution $u_h^a(x)$, obtained for problem (1) by the finite-difference method (from ([1],[2])), and passes to the limit first as $h \to 0$, and then as $a \to +\infty$, then in the limit one obtains the solution $u(x)$ of problem (3). However, in practice this is not feasible (passing to the limit $h \to 0$). In order that the method under discussion be suitable for

of the real solution of problem (3), one must investigate for what positive h and finite a the solution $u_h^a(x)$ differs from the exact solution $u(x)$ by an arbitrarily prescribed quantity ε (in one norm or another). For this it is necessary to obtain estimates of the deviations of $u_h^a(x)$ from $u^a(x)$ and of $u^a(x)$ from $u(x)$. The present work is devoted to this.

By the method indicated in ([3]), for the solution of problem (1) the inequality is established

$$\|u^a\|^2_{W_2^1(\Omega_1)} + a\|u^a\|^2_{W_2^1(\Omega_2)} \leq C_2\|f\|^2_{L_2(\Omega_1)}, \tag{4}$$

where C_2 depends only on the form of the domains Ω_1 and Ω_2.

From inequality (4) and the maximum principle it follows that $u^a(x)$ tends to $u(x)$ as $a \to \infty$ in such a way that

$$\|u(x) - u^a(x)\|_{C_{0,\alpha}(\Omega_1)} \leq \frac{C_3\|f\|_{L_2(\Omega_1)}}{\sqrt{a}}, \tag{5}$$

where $C_{0,\alpha}(\Omega_1)$ is the Hölder space with arbitrary exponent $\alpha < 1$; C_3 depends only on the form of the domains Ω_1, Ω_2 and on α.

For problem (1) the following difference scheme is constructed (see ([1])) with step $\Delta x_1 = \Delta x_2 = h$:

$$(a_h \bar{u}^a_{hx_1})_{\bar{x}_1} + (a_h \bar{u}^a_{hx_2})_{\bar{x}_2} = f_h(x_{ij});$$

$$a_h = \begin{cases} 1, & x_{ij} \in \Omega_1, \\ \dfrac{a+1}{2}, & x_{ij} \in S_1, \\ a, & x_{ij} \in \Omega_2; \end{cases} \qquad f_h(x_{ij}) = \frac{1}{h^2} \int_{x_1^i}^{x_1^i+h} \int_{x_2^j}^{x_2^j+h} f'(x_1, x_2)\, dx; \quad (6)$$

$$u_h^a\big|_{x_{ij}\in S_2} = 0.$$

Here x_{ij} denotes a grid node with coordinates $x_1 = x_1^i$, $x_2 = x_2^j$; $u^a_{hx_1}$, $u^a_{h\bar{x}_2}$, $u^a_{hx_2}$ are difference quotients of the sought grid function u_h^a ([1]).

It follows from ([2]) that the solution of problem (6) exists, is unique, and its multilinear interpolations $(u_h^a)'$ (see ([1])) converge weakly in the norm W_2^1 to the exact solution of problem (1).

In ([2]) the following estimate is also derived for u_h^a:

$$\int_\Omega \sum_{i=1}^{2} (\tilde{u}^a_{hx_i})^2\, dx \le C_4 \int_\Omega f^2(x_1, x_2)\, dx, \quad (7)$$

where $\tilde{u}^a_{hx_i}$ are the piecewise-constant interpolations ([2]) of the function $u^a_{hx_i}$; C_4 depends only on the dimensions of the rectangle Ω.

For the solution of problem (6) one succeeds in establishing that its piecewise-constant interpolations $\tilde{u}_h^a$ converge to the exact solution of problem (1) as $h \to 0$ in such a way that

$$\int_\Omega \left[(u - \tilde{u}_h^a)^2 + \sum_{i=1}^{2} \left(\frac{\partial u^a}{\partial x_i} - \tilde{u}^a_{hx_i} \right)^2 \right] dx \le C_5 \int_\Omega (f - \tilde{f}_h)^2\, dx + C_6 a h^{(1-\delta)/2} \|f\|^2_{L_2(\Omega_1)}, \quad (8)$$

where C_5 and C_6 depend only on the form of the domains Ω_1, Ω_2 and $0 < \delta < 1$.

Estimate (8) is obtained by comparing the integral identities for the solutions of problem (1), $u^a(x)$, and problem (6), u_h^a (see ([1])):

$$\int_\Omega a' \sum_{i=1}^{2} \frac{\partial u^a}{\partial x_i} \frac{\partial \Phi}{\partial x_i}\, dx = \int_\Omega f' \Phi\, dx, \quad (9)$$

where $\Phi \in \overset{0}{D}(\Omega)$ (see (9)) and

$$\int_\Omega \tilde{a}_h \sum_{i=1}^{2} \tilde{u}^a_{hx_i} \tilde{\Phi}_{hx_i}\, dx = \int_\Omega \tilde{f}_h \tilde{\Phi}_h\, dx, \tag{10}$$

where Φ_h is a grid function equal to zero at the nodes lying on S_2. In (9) it is assumed that $\Phi \equiv u^a - (u^a_h)'$. The values $u^a - (u^a_h)'$, $(u^a)'$ are fixed at the grid nodes. The resulting grid functions, whose piecewise-constant completions are denoted respectively by $\tilde{u}^a - (\tilde{u}^a_h)'$, $\tilde{u}^a$. Obviously, $\tilde{u}^a - (\tilde{u}^a_h)' \equiv \tilde{u}^a - \tilde{u}^a_h$. In (9), a', $\partial u^a/\partial x_i$, $u^a - (u^a_h)'$, $\partial(u^a - (u^a_h)')/\partial x_i$, $f'(x)$ are replaced respectively by $\tilde{a}_h$, $\tilde{u}^a_{x_i}$, $\tilde{u}^a - \tilde{u}^a_h$, $\tilde{u}^a_{x_i} - \tilde{u}^a_{hx_i}$, $\tilde{f}_h$. The arising residuals R_h are transferred to the right-hand side.

If in (10) we put $\tilde{\Phi}_h \equiv \tilde{u}^a - \tilde{u}^a_h$ and subtract (10) from the transformed identity (9), then one can easily derive

$$\int_\Omega \sum_{i=1}^{2} \left(\tilde{u}^a_{x_i} - \tilde{u}^a_{hx_i}\right)^2 dx \le |R_h|. \tag{11}$$

Here the following expressions will enter R_h as summands:

$$I_1 \equiv \int_\Omega \tilde{a}_h \sum_{i=1}^{2} \tilde{u}^a_{x_i} \left(\tilde{u}^a_{x_i} - \frac{\partial u^a}{\partial x_i}\right) dx,$$

$$I_2 \equiv -\int_\Omega \tilde{a}_h \sum_{i=1}^{2} \tilde{u}^a_{x_i} \left(\tilde{u}^a_{hx_i} - \frac{\partial (u^a_h)'}{\partial x_i}\right) dx,$$

$$I_3 \equiv \int_\Omega \tilde{a}_h \sum_{i=1}^{2} \left(\tilde{u}^a_{x_i} - \frac{\partial u^a}{\partial x_i}\right)\left(\frac{\partial u^a}{\partial x_i} - \frac{\partial (u^a_h)'}{\partial x_i}\right) dx,$$

$$I_4 \equiv \int_{\Omega_{he}} (a' - \tilde{a}_h) \sum_{i=1}^{2} \frac{\partial u^a}{\partial x_i} \left(\frac{\partial u^a}{\partial x_i} - \frac{\partial (u^a_h)'}{\partial x_i}\right) dx$$

(Ω_{he} is the aggregate of grid squares through which the line of discontinuity of a' passes).

By applying the Cauchy inequality, the embedding theorems, and estimates (4), (7), it is shown that I_1, I_3, I_4 can be estimated from above by the quantity

$$aC_7 \left(\int_\Omega \sum_{i=1}^{2} \left(\frac{\partial u^a}{\partial x_i} - \tilde{u}^a_{x_i}\right)^2 dx\right)^{1/2} \|f\|_{L_2(\Omega_1)}, \tag{12}$$

where C_7 is a constant depending only on the form of the domains Ω_1 and Ω_2.

One can arrive at the same conclusion also with respect to I_2, after first representing $\partial(u_h^a)'/\partial x_i$ in it through the difference quotients $u_{hx_i}^a$, $u_{hx_1x_2}^a$, and then transferring, by summation by parts, the difference with respect to x_i from $u_{hx_1x_2}^a$ to $u_{x_j}^a$.

The estimate of

$$\int_\Omega \sum_{i=1}^{2} \left(\frac{\partial u^a}{\partial x_i} - \tilde{u}_{x_i}^a \right)^2 dx$$

is carried out by splitting it into three parts:

$$\int_\Omega \sum_{i=1}^{2} \left(\frac{\partial u^a}{\partial x_i} - \tilde{u}_{x_i}^a \right)^2 dx = \sum_{k=1}^{2} \int_{\Omega_{hk}} \sum_{i=1}^{2} \left(\frac{\partial u^a}{\partial x_i} - \tilde{u}_{x_i}^a \right)^2 dx + \int_{\Omega_{he}} \sum_{i=1}^{2} \left(\frac{\partial u^a}{\partial x_i} - \tilde{u}_{x_i}^a \right)^2 dx.$$

Ω_{h1} and Ω_{h2} are the aggregates of grid squares respectively from the domains

Ω_1 and Ω_2, which are not intersected by the line S_1. The function $\tilde{u}_{x_1}^a(x)$, for $x = (x_1, x_2)$ varying in the square $x_1^l \le x_1 \le x_1^l + h,\ x_2^m \le x_2 \le x_2^m + h$, is the average of the function $\partial u^a/\partial x_1$ along the line $x_2 = x_2^m,\ x_1^l \le x_1 \le x_1^l + h$:

$$\tilde{u}_{x_1}^a(x) = \frac{1}{h} \int_{x_1^l}^{x_1^l + h} \frac{\partial u^a(\xi, x_2^m)}{\partial \xi}\, d\xi.$$

Similarly for $\tilde{u}_{x_2}^a$. Comparing $\partial u^a/\partial x_i$ with its average and using the embedding theorems and estimates (4), one can obtain

$$\sum_{k=1}^{2} \int_{\Omega_{hk}} \sum_{i=1}^{2} \left(\frac{\partial u^a}{\partial x_i} - \tilde{u}_{x_i}^a \right)^2 dx \le C_8 h^2 \|f\|^2_{L_2(\Omega_1)}, \tag{13}$$

where C_8 depends only on the form of the domains Ω_1 and Ω_2.

From the embedding theorems and the fact that the total area of the squares contained in Ω_{he} does not exceed kh, where $k > 0$ depends only on the form of S_1, it follows that

$$\int_{\Omega_{he}} \sum_{i=1}^{2} \left(\frac{\partial u^a}{\partial x_i} - \tilde{u}_{x_i}^a \right)^2 dx \le C_9 h^{1-\delta} \|f\|^2_{L_2(\Omega_1)}, \tag{14}$$

where C_9 depends only on the form of the domains Ω_1, Ω_2, and $0 < \delta < 1$.

From (11), (12), (13), (14) there follows estimate (8), which, together with estimate (5), makes it possible to solve problem (3) for an arbitrary domain.

Results similar to those obtained in the present work can be generalized to the three-dimensional case and to more complicated problems with discontinuous coefficients.

Recently a paper by V. K. Saul' ev ([4]) was published, in which similar questions are discussed.

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Note: Figure translations are in progress. See original paper for figures.