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Abstract

Full Text

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ON THE SUM OF THE DIFFERENCES OF THE EIGENVALUES OF TWO SELF-ADJOINT OPERATORS

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1. The formula for the sum of the differences of the eigenvalues of two regular Sturm–Liouville operators was first obtained in the work of I. M. Gelfand and B. M. Levitan ([1]), and then by other methods in the works of L. A. Dikii ([2]) and C. Halberg and V. Kramer ([3]). In the present paper an analogue of the aforementioned formula is derived for the sum of the differences of the eigenvalues of two singular Sturm–Liouville operators differing from one another by a finite perturbation. The proposed method is based on certain inequalities obtained in the present work for the sums

$$\sum_{n=1}^{N}(\mu_n - \lambda_n),$$

where $\mu_1 \leqslant \mu_2 \leqslant \cdots$ and $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$ are the eigenvalues of self-adjoint operators C and A, bounded below, in a Hilbert space H, with the same domain of definition D_A, and on the asymptotics of the spectral function of a Sturm–Liouville operator found by B. M. Levitan ([4]). In addition, with the aid of the inequalities found, some abstract theorems are proved. In particular, it is shown that if the difference

$$C - A = B$$

is a nuclear operator, then the series

$$\sum_{n=1}^{\infty}(\mu_n - \lambda_n)$$

converges to the trace of the operator B.

2. Let us consider in a Hilbert space H a self-adjoint operator A with domain of definition D_A, bounded below, with eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$ and the corresponding orthonormal eigenfunctions $\varphi_1, \varphi_2, \ldots$, and any orthonormal basis $f_n \in D_A,\ n = 1, 2, \ldots$. Then the following holds:

Lemma 1.

$$\sum_{n=1}^{N}(Af_n, f_n) \geqslant \sum_{n=1}^{N}\lambda_n. \tag{1}$$

Proof. It is obvious that

$$f_n = \sum_{k=1}^{\infty} u_{nk}\varphi_k,$$

where $\|u_{nk}\|$ is a unitary matrix. Then

$$Af_n = \sum_{k=1}^{\infty} \lambda_k u_{nk}\varphi_k.$$

Hence

$$(Af_n, f_n) = \sum_{k=1}^{\infty} \lambda_k |u_{nk}|^2.$$

Thus,

$$\begin{aligned}
\sum_{n=1}^{N}(Af_n, f_n) &= \sum_{n=1}^{N}\sum_{k=1}^{\infty}\lambda_k|u_{nk}|^2 \\
&= \sum_{n=1}^{N}\sum_{k=1}^{N}\lambda_k|u_{nk}|^2 + \sum_{n=1}^{N}\sum_{k=N+1}^{\infty}\lambda_k|u_{nk}|^2 \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{N}\lambda_k|u_{nk}|^2 - \sum_{n=N+1}^{\infty}\sum_{k=1}^{N}\lambda_k|u_{nk}|^2 + \sum_{n=1}^{N}\sum_{k=N+1}^{\infty}\lambda_k|u_{nk}|^2 \\
&\geq \sum_{k=1}^{N}\lambda_k + \lambda_{N+1}\left\{\sum_{n=1}^{N}\sum_{k=N+1}^{\infty}|u_{nk}|^2 - \sum_{n=N+1}^{\infty}\sum_{k=1}^{N}|u_{nk}|^2\right\} = \sum_{k=1}^{N}\lambda_k.
\end{aligned}$$

Thus, the lemma is proved.

Let us note that an inequality similar to inequality (1), for finite matrices, was proved by Ky Fan ([5]).

Corollary 1. If $\omega(t)$ is a continuous nondecreasing function, then

$$\sum_{n=1}^{N}(\omega(A)f_n, f_n) \geq \sum_{n=1}^{N}\omega(\lambda_n). \tag{2}$$

This corollary is proved analogously to Lemma 1.

Let C be a self-adjoint operator in H, bounded from below, with domain of definition $D_C = D_A$, with eigenvalues $\mu_1 \leq \mu_2 \leq \cdots$ and corresponding orthonormal eigenfunctions $\psi_1, \psi_2, \ldots$. From Lemma 1 the inequalities follow

$$\sum_{n=1}^{N}(A\psi_n, \psi_n) \geq \sum_{n=1}^{N}\lambda_n, \qquad \sum_{n=1}^{N}(C\varphi_n, \varphi_n) \geq \sum_{n=1}^{N}\mu_n. \tag{3}$$

Theorem 1. Let $B = C - A$. Then

$$\sum_{n=1}^{N}(B\psi_n, \psi_n) \leq \sum_{n=1}^{N}(\mu_n - \lambda_n) \leq \sum_{n=1}^{N}(B\varphi_n, \varphi_n). \tag{4}$$

Proof. Using inequalities (3), we obtain

$$\sum_{n=1}^{N}(B\psi_n, \psi_n) = \sum_{n=1}^{N}\{(C\psi_n, \psi_n) - (A\psi_n, \psi_n)\} \leq \sum_{n=1}^{N}(\mu_n - \lambda_n) \leq$$

$$\leq \sum_{n=1}^{N}\{(C\varphi_n, \varphi_n) - (A\varphi_n, \varphi_n)\} = \sum_{n=1}^{N}(B\varphi_n, \varphi_n),$$

which proves the theorem.

Using Corollary 1, one can prove the following theorem.

Theorem 2. Let $\omega(t)$ be a continuous nondecreasing function and $B_\omega = \omega(C) - \omega(A)$. Then

$$\sum_{n=1}^{N}(B_\omega\psi_n, \psi_n) \leq \sum_{n=1}^{N}\{\omega(\mu_n) - \omega(\lambda_n)\} \leq \sum_{n=1}^{N}(B_\omega\varphi_n, \varphi_n). \tag{5}$$

Definition. If for an orthonormal basis $\{f_n\}$ there exists the limit $\lim\limits_{N\to\infty}\sum_{n=1}^{N}(Af_n, f_n)$, then this limit is called the **trace of the operator A in the basis $\{f_n\}$**.

Using inequalities (4) and (5), we obtain the following theorem.

Theorem 3. If the traces of the operator $B = C - A$ in the bases $\{\varphi_n\}$ and $\{\psi_n\}$ are equal, then

$$\lim_{N\to\infty}\sum_{n=1}^{N}(\mu_n - \lambda_n) = \lim_{N\to\infty}\sum_{n=1}^{N}(B\varphi_n, \varphi_n). \tag{6}$$

Theorem 4. If the traces of the operator $B_\omega = \omega(C) - \omega(A)$, where $\omega(t)$ is a continuous nondecreasing function, in the bases $\{\varphi_n\}$ and $\{\psi_n\}$ are equal, then

$$\lim_{N\to\infty}\sum_{n=1}^{N}\{\omega(\mu_n) - \omega(\lambda_n)\} = \lim_{N\to\infty}\sum_{n=1}^{N}(B_\omega\varphi_n, \varphi_n). \tag{7}$$

It is known that a nuclear operator has equal traces in any basis. Therefore the following theorems hold:

Theorem 5. Let A be any self-adjoint operator in H, bounded below and with discrete spectrum, and let B be a nuclear self-adjoint operator. Then the series of differences of the eigenvalues of the operators $C = A + B$ and A converges to the trace of the operator B.

Theorem 6. If $\omega(t)$ is a continuous nondecreasing function and $\omega(C) - \omega(A) = B_\omega$ is a nuclear operator, then

$$\lim_{N\to\infty}\sum_{n=1}^{N}\{\omega(\mu_n) - \omega(\lambda_n)\}$$

exists and is equal to the trace of the operator B_ω.

We note that in Theorem 6 it is not required that the operator $B = C - A$ be nuclear. For example, if A and C are of Hilbert-Schmidt type and $\omega(t) \geq t^2$, then the conditions of the theorem are satisfied.

3. With the aid of the results obtained above, one can compute the sum of differences of eigenvalues of the operators L_0 and L generated by the differential equations

$$y'' + \{\lambda - q(x)\}y = 0, \tag{8}$$

$$y'' + \{\lambda - q(x) - p(x)\}y = 0, \tag{9}$$

respectively. Let us consider the following examples.

A. Let $-\infty < x < \infty$, let the continuous function $q(x) \to \infty$ as $|x| \to \infty$, and let $p(x)$ be a finite continuous function. Then the spectra of the operators L_0 and L are discrete. Denote the eigenvalues of the operators L_0 and L by $\lambda_1 < \lambda_2 < \cdots$ and $\mu_1 < \mu_2 < \cdots$, and the eigenfunctions by $\varphi_1(x), \varphi_2(x), \ldots$ and $\psi_1(x), \psi_2(x), \ldots$, respectively. Then the following holds:

Theorem 7. If $\int p(x)\,dx = 0$, then the series of differences of the eigenvalues of the operators L and L_0 converges and is equal to zero.

Proof. $D_{L_0} = D_L$, therefore, according to Theorem 3, it suffices to show that

$$\lim_{N\to\infty}\sum_{n=1}^{N}\int p(x)\psi_n^2(x)\,dx = \lim_{N\to\infty}\sum_{n=1}^{N}\int p(x)\varphi_n^2(x)\,dx.$$

We note that

$$\lim_{N\to\infty}\sum_{n=1}^{N}\int p(x)\psi_n^2(x)\,dx = \lim_{\lambda\to\infty}\sum_{\mu_n<\lambda}\int p(x)\psi_n^2(x)\,dx.$$

On the other hand, it is known ([4]) that in any bounded interval of variation of x the formula

$$\sum_{\mu_n<\lambda}\psi_n^2(x) = \frac{2}{\pi}\sqrt{\lambda} + o(1)$$

holds. Further, since $p(x)$ is a finite function and $\int p(x)\,dx = 0$, we have

$$\lim_{\lambda\to\infty}\sum_{\mu_n<\lambda}\int p(x)\psi_n^2(x)\,dx = 0.$$

It is proved in a completely analogous way that

$$\lim_{\lambda\to\infty}\sum_{\lambda_n<\lambda}\int p(x)\varphi_n^2(x)\,dx = 0.$$

This also proves the theorem.

B. Let the operators L_0 and L be generated by equations (8) and (9), respectively, given on the half-axis $0 \le x < \infty$, with the boundary condition $y'(0) - hy(0) = 0$, where h is a real number. Then, if a continuous function $q(x) \to \infty$ as $x \to \infty$, and $p(x)$ is a finite continuous function, the operators L_0 and L have discrete spectra $\lambda_1 < \lambda_2 < \ldots$ and $\mu_1 < \mu_2 < \ldots$, and eigenfunctions $\varphi_1(x), \varphi_2(x), \ldots$ and $\psi_1(x), \psi_2(x), \ldots$, respectively.

Theorem 8. *If the function $p(x)$ has a first derivative in a small neighborhood of zero and*

$$\int p(x)\,dx = 0,$$

then

$$\lim_{N\to\infty}\sum_{n=1}^{N}(\mu_n - \lambda_n) = \frac{p(0)}{4}. \tag{10}$$

This formula is an analogue of the formula of I. M. Gelfand and B. M. Levitan [1] (they considered the regular case).

Proof. Obviously, it is sufficient to show that

$$\lim_{\lambda\to\infty}\sum_{\mu_n<\lambda}\int p(x)\psi_n^2(x)\,dx-\lim_{\lambda\to\infty}\sum_{\lambda_n<\lambda}\int p(x)\varphi_n^2(x)\,dx=\frac{p(0)}{4}. \tag{11}$$

It is known [4] that for large λ

$$\sum_{\lambda_n<\lambda}\varphi_n^2(x)-\theta(x,x,\lambda)=o(1),\qquad \text{where } \theta(x,x,\lambda)=\frac{2}{\pi}\int_0^{\sqrt{\lambda}}\cos^2 sx\,ds.$$

Therefore

$$\lim_{\lambda\to\infty}\sum_{\lambda_n<\lambda}\int p(x)\varphi_n^2(x)\,dx=\lim_{\lambda\to\infty}\int p(x)\theta(x,x,\lambda)\,dx. \tag{12}$$

Let us compute the last limit. First note that

$$\theta(x,x,\lambda)=\frac{1}{\pi}\sqrt{\lambda}+\frac{1}{2\pi}\frac{\sin 2\sqrt{\lambda}\,x}{x}. \tag{13}$$

Therefore

$$\lim_{\lambda\to\infty}\int p(x)\theta(x,x,\lambda)\,dx=\frac{1}{2\pi}\lim_{\lambda\to\infty}\int p(x)\frac{\sin 2\sqrt{\lambda}\,x}{x}\,dx=\frac{p(0)}{4}. \tag{14}$$

Similarly one can show that

$$\lim_{\lambda\to\infty}\sum_{\mu_n<\lambda}\int p(x)\psi_n^2(x)\,dx=\frac{p(0)}{4}. \tag{15}$$

The theorem is proved.

In proving Theorems 7 and 8, the following was proved along the way.

Theorem 9. *In cases A and B, for the convergence of the series*

$$\sum_{n=1}^{N}(\mu_n-\lambda_n)$$

it is necessary and sufficient that

$$\int p(x)\,dx = 0.$$

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Note: Figure translations are in progress. See original paper for figures.