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Abstract

Full Text

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THE PICARD THEOREM FOR ENTIRE BIANALYTIC FUNCTIONS

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1. According to the well-known Picard theorem, every entire analytic function $\varphi(z)$, distinct from a constant, assumes every finite value w, with possibly one exception. Consider the class of entire bianalytic functions, i.e., functions of the form

$$B(z) = \varphi(z) + \bar{z} \cdot \psi(z), \tag{1}$$

where $\varphi(z)$ and $\psi(z)$ are entire analytic functions, and $\bar{z}$ is the number conjugate to z.

A function of the form (1) may omit infinitely many values and at the same time not be constant. Thus, for example, the function $w = z(4-\bar{z})$ assumes no value $w = u+iv$ for which $v^2 > 16(4-u)$; the function $w = z^2+|z|^2$ assumes no value belonging to the imaginary axis (except $w = 0$); the functions $A + B(z + e^{i\alpha}\bar{z})$, $w = A+B|z+c|^2$ map the entire z-plane onto a straight line or a ray (A, B, c, α are constants, $\operatorname{Im}\alpha = 0$).

However, these cases are exceptional: under quite simple additional restrictions for functions (1), the Picard theorem holds (or even stronger results hold). The theory of normal families is the principal tool in the following arguments.

2. Below the notation $\psi(z) \not\equiv 0$ means: $\psi(z)$ is not identically zero.

Lemma 1. *If the function $\psi(z)$ is entire analytic, $\psi(z) \not\equiv 0$, and $\psi(z)$ has at least one zero, then the entire bianalytic function of the form*

$$f(z) = \varphi(z) + |z - z_1|^2\psi(z) \tag{2}$$

assumes every (without exception) complex value a.

Proof. It suffices to restrict ourselves to the case $a = 0$, $z_1 = 0$. Choose a sequence $\{c_n\}$ $(n = 0, 1, 2, ...)$ so that 1) $c \equiv c_0 > 0$; $c_n > c_{n-1}$; $c_n \to \infty$ as $n \to \infty$; 2) $\psi(z) \neq 0$ on $\Gamma_n\{|z| = c_n\}$; 3) $\psi(z)$ has at least one zero (z_0) in $D\{|z| < c\}$.

Consider the family of functions

$$F_n(z) = \varphi(z) + c_n^2 \psi(z).$$

It is quasinormal in D (cf. (^{1}), p. 61). The irregular point will be the point z_0. In any (small) neighborhood δ of the point z_0 the function $F_n(z)$ (beginning with some number N) assumes the value $a = 0$ (cf. (^{1}), p. 62). Therefore, for $n > N$,

$$\frac{1}{2\pi} \Delta_{\Gamma_n} \arg F_n(z) \geqslant 1.$$

But on Γ_n, $f(z) \equiv F_n(z)$. Therefore

$$\frac{1}{2\pi} \Delta_{\Gamma_n} \arg f(z) \geqslant 1,$$

so that $f(z)$ vanishes at least at one point inside Γ_n (see (^{2})).

Corollary. If $\psi(z)$ is an entire analytic function, $\psi(z) \not\equiv 0$, and $\psi(z)$ has at least two zeros (or at least one, but of order $p \geqslant 2$), then the entire bianalytic function (1) assumes every complex value.

Indeed, in this case it is possible to write (1) in the form of the function (2), satisfying the conditions of Lemma 1.

Lemma 2. If the entire analytic function $\varphi(z)$ is not a polynomial of second (or lower) degree, then the function

$$f(z) = \varphi(z) + |z|^2 \tag{3}$$

assumes every complex value a.

Proof. It suffices to restrict ourselves to the case $a = 0$. Suppose that (3) has no zeros. Then for every $c > 0$ on the circle $\Gamma\{|z| = c\}$

$$\frac{1}{2\pi} \Delta_{\Gamma} \arg f(z) = 0. \tag{4}$$

Let

$$\Phi_c(z) = \varphi(z) + c^2. \tag{5}$$

On Γ, $\Phi_c(z) \equiv f(z)$. Therefore

$$\frac{1}{2\pi} \Delta_{\Gamma} \arg \Phi_c(z) = 0, \tag{6}$$

i.e. $\Phi_c(z)$ does not assume the value 0 inside Γ, for any $c > 0$. It is easy to verify that $\Phi_c(z)$ does not assume inside Γ any negative value d either—in the contrary case, inside the circle $\Gamma_1\{|z| = c_1 = \sqrt{c^2 + |d|} > c\}$ the function $\Phi_{c_1}(z) \equiv \varphi(z) + c_1^2$ would have at least one zero, contrary to (6).

Take an arbitrary sequence $\{c_n\}$ such that $c_n > 0$, $c_{n+1} > c_n$, $c_n \to \infty$. Consider in $D\{|z| < 1\}$ the family of functions

$$F_n(z) = 1 + \varphi(c_{nz})/c_n^2. \tag{7}$$

None of them assumes in D the values 0 and -1, so that the family (7) must be normal in D (see (1), p. 58). Hence from it one can extract a sequence $\{F_{n_k}(z)\}$ which must converge uniformly inside D either to the identically infinite function, or to an analytic function. But the first is impossible, since as $n \to \infty$, $F_n(0) \to 1$ (see (7)).

Let us show that the second is also impossible. Let $\varphi(z) = a_0 + a_1 z + a_2 z^2 + z^2\theta(z)$, where $\theta(z)$ is an entire function, $\theta(0) = 0$, $\theta(z) \not\equiv 0$,

$$F_n(z) = 1 + \frac{a_0}{c_n^2} + \frac{a_1}{c_n} z + a_2 z^2 + z^2 \theta(c_{nz}). \tag{8}$$

Denote by $M(R)$ the maximum of $|\theta(w)|$ on the circle $|w| = R$. Choose on the circle $|z| = \frac{1}{2}$ points $z_n = \frac{1}{2}\exp(i\alpha_n)$ so that

$$|\theta(c_{nz}n)| = \left|\theta\left[\frac{1}{2}c_n \exp(i\alpha_n)\right]\right| = M\left(\frac{1}{2}c_n\right).$$

It is clear that as $n \to \infty$, $M\left(\frac{1}{2}c_n\right) \to \infty$ (for otherwise, by Liouville' s theorem, $\theta(z) \equiv \text{const}$, and $\varphi(z)$ would be a polynomial of degree not exceeding two). Therefore $F_n(z_n) \to \infty$, so that $\{F_{n_k}(z)\}$ cannot converge uniformly inside D to an analytic function.

Thus the family (7) cannot be normal in D—contrary to the conclusion obtained earlier. From this contradiction the validity of Lemma 2 follows.

Lemma 3. In order that the entire bianalytic function

$$B(z) = \Phi(z) + \bar{z} \tag{9}$$

($\Phi(z)$ is an analytic function) take every complex value, it is necessary and sufficient that it not have the form

$$a_0 + e^{i\alpha} z + \bar{z} \quad (a_0 = \text{const},\ \alpha = \text{const},\ \operatorname{Im}\alpha = 0). \tag{10}$$

Proof. Case 1. $\Phi(z)$ is a transcendental function. If $B(z)$ never takes some value a, then on any circle $\Gamma\{|z| = c > 0\}$

$$\frac{1}{2\pi}\Delta_\Gamma \arg[B(z)-a] = 0.$$

Consider the function $f(z) \equiv z[B(z)-a] \equiv \varphi(z)+|z|^2$ (here $\varphi(z) = z[\Phi(z)-a]$). It is clear that (for any $c > 0$)

$$\frac{1}{2\pi}\Delta_\Gamma \arg f(z) = 1. \tag{11}$$

We retain the same notation as in the proof of Lemma 2, but now choose the numbers c_n in the following way. Since $\theta(z)$ in the present case is an entire transcendental function, there exists a sequence of points

$$w_n = \frac{1}{2}c_n \exp(i\alpha_n), \tag{12}$$

such that $c_n > 0$, $c_{n+1} > c_n$, $c_n \to \infty$, $\theta(w_n) = \pm 1$ $(n = 1, 2, ...)$. We shall assume that precisely these numbers c_n occur in (7).

Arguing in the same way as in the proof of Lemma 2, it is possible to show on the basis of (11) that each function (7) assumes in $D\{|z| < 1\}$ the values 0 and -1 no more than once each. Therefore (see (^{1}), p. 63) the family of functions (7) is quasinormal in D of order not exceeding 1. Suppose first that the order of quasinormality of the family (7) is equal to 1. Then there exists a sequence $\{F_{n_k}(z)\}$ which in fact has an irregular point (and, moreover, only one). But $\{F_{n_k}(z)\}$ has a finite limit at the point $z = 0$. Therefore (cf. (^{1}), pp. 61-62) precisely the point $z = 0$ will be irregular for this sequence. Surround this point by the disk $\delta\{|z| < \varepsilon < 1/4\}$. Beginning with some number N, all functions $F_{n_k}(z)$ assume inside δ the values 0 and -1. But each of these functions assumes these values no more than once each, so that in the annulus $G\{\varepsilon < |z| < 1\}$ the sequence (7) must form a normal family. But this is impossible. Indeed, on the circle $|z| = 1/2$ there exist points $z_n = \frac{1}{2}\exp(i\alpha_n)$ such that, for every n, $\theta(c_{nz}n) = \pm 1$. The set $\{F_n(z_n)\}$ is bounded, so that no subsequence $\{F_{n_k}(z)\}$ can converge uniformly inside G to infinity. It is evident that this subsequence also cannot have a finite limit.

Case 2. $\Phi(z)$ is a polynomial of degree n. If $n > 1$, then for sufficiently large c, on the circle $\Gamma\{|z| = c\}$

$$|\Phi(z)| > |\bar{z}|, \qquad \frac{1}{2\pi}\Delta_\Gamma \arg\Phi(z) = n.$$

By the generalized Rouché theorem (see, for example, (^{2}))

$$\frac{1}{2\pi}\Delta_\Gamma \arg f(z) = \frac{1}{2\pi}\Delta_\Gamma \arg[\Phi(z)+\bar{z}] = \frac{1}{2\pi}\Delta_\Gamma \arg\Phi(z) = n > 1,$$

so that $f(z)$ has at least one zero inside Γ (see (^{2})).

If $n = 1,\ \Phi(z) = Az+B,\ A \neq e^{i\alpha}$, then the zero of the function (9) is computed directly.

If the function $B(z)$ has the form (10), then it assumes every value w belonging to the line

$$w = a_0 + \exp\left(\frac{1}{2}i\alpha\right) t \tag{13}$$

at infinitely many points (the latter fill out an entire line), and does not assume at all values not belonging to the line (13).

Lemma 3 could also have been proved by relying on a generalization of the well-known theorem of E. Landau given by L. Bieberbach. As S. Ts. Sarkisyan has informed me, he has found an entirely different method for proving Lemma 3.

Remark. Refining the preceding arguments, it is possible to prove:

1) If $\psi(z)$ has infinitely many zeros, then each of the entire bianalytic functions (1) and (2) assumes every complex value at infinitely many points.
2) If $\varphi(z)$ and $\Phi(z)$ are entire transcendental functions, then each of the functions (3) and (9) assumes every complex value at infinitely many points.

A simple consequence of Lemmas 1–3 is the following

Theorem (Picard theorem for bianalytic functions). *An entire bianalytic function (1) which fails to assume any two values is a polynomial of degree not exceeding two with respect to the pair of variables z and $\bar{z}$, i.e. has the form*

$$B(z) = a_0 + a_1 z + a_2 z^2 + (b_0 + b_1 z)\bar{z},$$

where a_0, a_1, a_2, b_0, b_1 are constants.

From Lemmas 1–3 one can derive certain corollaries for meromorphic bianalytic functions. We give here two such corollaries.

I. If $R(z)$ is a meromorphic analytic function having at least two poles (or at least one pole of order not less than two), then the meromorphic bianalytic function $R(z) + \bar{z}$ assumes **every** complex value.

II. If $R(z)$ is a meromorphic function, but not a polynomial of second (or lower) degree, then $R(z) + |z|^2$ assumes **every** complex value.

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Note: Figure translations are in progress. See original paper for figures.

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