MATHEMATICS

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Abstract

Full Text

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ON ORTHOGONAL SERIES NOT SUMMABLE BY LINEAR METHODS

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Let $\{\varphi_n(x)\}$ be a system of functions orthonormal on $[a, b]$, and let the coefficients of the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \tag{1}$$

satisfy the condition $\sum c_n^2 < \infty$. With the aid of the triangular matrix

$$\lambda = \{\lambda_k^n\} \quad (k = 0, 1, \dots, n+1;\ n = 0, 1, \dots;\ \lambda_0^n = 1,\ \lambda_{n+1}^n = 0) \tag{2}$$

to each series (1) there is put in correspondence the sequence of λ-means

$$U_n(x, \{\varphi\}, \lambda) = \sum_{k=0}^{n} \lambda_k^n c_k \varphi_k(x) \quad (n = 0, 1, \dots).$$

If the matrix (2) satisfies the conditions

$$\text{a)} \lim_{n\to\infty} \lambda_k^n = 1 \quad (k = 1, 2, \dots), \qquad \text{b)} \sum_{k=0}^{n} |\Delta\lambda_k^n| \le A, \tag{3}$$

where $\Delta\lambda_k^n = \lambda_k^n - \lambda_{k+1}^n$, and A is a constant independent of n, then the summation method (λ) determined by this matrix is regular in the sense of Toeplitz. Put

$$L_n(x, \{\varphi\}) = \int_a^b \left| \sum_{k=0}^{n} \varphi_k(x)\varphi_k(t) \right| dt \quad (n = 0, 1, \dots),$$

$$L_n(x, \{\varphi\}, \lambda) = \int_a^b \left| \sum_{k=0}^{n} \lambda_k^n \varphi_k(x)\varphi_k(t) \right| dt \quad (n = 0, 1, \dots).$$

P. L. Ul' yanov posed the following question: can a nondecreasing majorant of the Lebesgue functions of some summation method (λ) always be taken as Weyl multipliers for the almost-everywhere summability of the series (1) by this method? (Cf. (^{1,2}); (^{3}), Theorem 5.5.5; ($^{4-6}$).) We give an answer to this question.

Theorem 1. *Let $f(x) \to \infty$ as $x \to \infty$. Then there exist a summation method (λ), regular in the sense of Toeplitz, and a system of functions $\{\psi_n(x)\}$ orthonormal on $[a, b]$, such that the Lebesgue functions of this method (λ) are bounded by a function $\gamma(n)$ increasing to infinity, i.e.*

$$L_n(x, \{\psi\}, \lambda) = O(\gamma(n)) \quad \text{for almost all } x \in (a, b) \tag{4}$$

and there exists an orthogonal series, not summable almost everywhere by the method (λ),

$$\sum_{n=0}^{\infty} b_n \psi_n(x), \tag{5}$$

coefficients of which satisfy the condition

$$\sum_{n=1}^{\infty} b_n^2 f[\gamma(n)] < \infty. \tag{6}$$

Theorem 2. For every Toeplitz-regular summability method (λ) there exists a system of functions $\{\psi_n(x)\}$, orthonormal on $[a, b]$, such that $L_n(x, \{\psi\})$ are bounded by a function $\nu(n)$ increasing strictly to infinity and, whatever increasing sequence $\{w(n)\}$ with the property $w(n) = o(\nu(n))$ may be, there exists an orthogonal series (5), nonsummable almost everywhere by the method (λ), the coefficients of which satisfy the condition

$$\sum_{n=1}^{\infty} b_n^2 w(n) < \infty. \tag{7}$$

Theorem 3. For every Toeplitz-regular summability method (λ) there exists a system of functions $\{\psi_n(x)\}$, orthonormal on $[a, b]$, such that the Lebesgue functions of this method $L_n(x, \{\psi\}, \lambda)$ are bounded by a function $\nu(n)$ increasing strictly to infinity and, whatever increasing sequence $\{w(n)\}$ with the property $w(n) = o(\nu(n))$ may be, there exists an orthogonal series (5), nonsummable almost everywhere by the method (λ), the coefficients of which satisfy condition (7).

Theorem 2 is an immediate consequence of Leindler' s theorem ((^7), theorem 3) and Kaczmarz' s theorem ((^3), theorem 5.7.4). Further, since from the estimate $L_n(x,\{\psi\}) = O(\nu(n))$, where $\nu(n)$ is nondecreasing, for any Toeplitz-regular method (λ) we have

$$L_n(x,\{\psi\},\lambda) = \int_a^b \left| \sum_{k=0}^{n} \Delta\lambda_k^n \sum_{j=0}^{k} \psi_j(x)\psi_j(t) \right| dt \le \sum_{k=0}^{n} |\Delta\lambda_k^n| L_k(x,\{\psi\}) = O(\nu(n)), \tag{8}$$

Theorem 3 is a consequence of Theorem 2.

Proof of Theorem 1. Let $\{m_i\}$ be an arbitrary sequence of indices for which $m_{i+1}/m_i > 1$ $(m_0 = 0)$. We choose a strictly increasing function $\nu(n)$ so that $\nu(m_i) = O(\log^2 i)$. We now define a function $p(i)$, increasing to infinity, from the condition $f[\nu(p(i))] = o(\nu(i))$, where $f(x) \to \infty$ as $x \to \infty$. For each $m \in [m_i, m_{i+1})$ we put

$$\lambda_k^m = \begin{cases} 0, & \text{if } k = m_j \text{ and } p(i)+1 \le j \le i, \\ \overline{\lambda}_k^m, & \text{for the remaining } k, \end{cases} \tag{9}$$

where

$$\overline{\lambda}_k^m = \begin{cases} 1, & \text{for } 0 \le k \le m_{p(i)}, \\ \dfrac{1}{l_i}\left(1 - \dfrac{k}{m_{i+1}}\right)^r, & \text{for } m_{p(i)}+1 \le k \le m, \end{cases} \tag{10}$$

$r \ge 1$ is a fixed number and

$$l_i = \sum_{j=p(i)}^{i} \left(1 - \frac{m_j}{m_{i+1}}\right)^r.$$

Since $\overline{\lambda}_k^m$ does not increase as k increases, for $m \in [m_i, m_{i+1})$

$$\sum_{k=0}^{m} |\Delta\lambda_k^m| \le \sum_{k=0}^{m} |\Delta\overline{\lambda}_k^m| + 2\sum_{j=p(i)+1}^{i} \overline{\lambda}_{m_j}^m \le 1 + \frac{2}{l_i} \sum_{j=p(i)+1}^{i} \left(1 - \frac{m_j}{m_{i+1}}\right)^r \le 3.$$

This means that the summation method (λ), defined by the sequence $\{\lambda_k^n\}$ given in (9) and (10), is regular in the sense of Toeplitz.

We now construct the system $\{\psi_n(x)\}$. Following Leindler ((7), Theorem 3), we shall apply Tandori' s theorem ([8]). Namely, since

$$v(m_i) = O(\log^2 i),$$

by Tandori' s theorem there exists an orthonormal system $\{\Phi_n(x)\}$ such that

$$L_n(x, \{\Phi\}) = O(v(n)) \tag{11}$$

and the series

$$\sum_{n=0}^{\infty} a_n \Phi_n(x) \tag{12}$$

diverges almost everywhere, although its coefficients satisfy the condition

$$\sum_{n=1}^{\infty} a_n^2 f[v(\rho(n))] < \infty. \tag{13}$$

We may assume that the functions $\Phi_n(x)$ satisfy the condition

$$\Phi_n\left(x + \frac{b-a}{2}\right) = \Phi_n(x) \quad \text{for } a < x < \frac{a+b}{2}.$$

Let $\chi_n(x)$ $(n = 1, 2, ...)$ be the Haar functions, $I = [u, v]$ an arbitrary interval, and

$$\chi_n(I; x) = \begin{cases} \chi_n\left(\dfrac{x-u}{v-u}\right), & \text{for } u < x < v, \\ 0, & \text{for } x \notin (u, v). \end{cases}$$

Denoting $I_1 = \left[a, \dfrac{a+b}{2}\right]$ and $I_2 = \left[\dfrac{a+b}{2}, b\right]$, set

$$\overline{\chi}_n(x) = \frac{1}{\sqrt{b-a}} \{\chi_n(I_1; x) - \chi_n(I_2; x)\}.$$

For the system $\{\overline{\chi}_n(x)\}$ the estimate ([9]) is known:

$$L_n(x, \{\overline{\chi}\}) = \int_a^b \left| \sum_{k=1}^{n} \overline{\chi}_n(x) \overline{\chi}_n(t) \right| dt = O(1). \tag{14}$$

Now put

$$\psi_k(x) = \begin{cases} \Phi_j(x), & \text{for } k = m_j, \\ \overline{\chi}_{k-j}(x), & \text{for } m_j < k < m_{j+1}. \end{cases} \tag{15}$$

It is evident that the system $\{\psi_n(x)\}$ is orthonormal. Let us estimate $L_m(x, \{\psi\}, \lambda)$. Let $m \in [m_i, m_{i+1})$. Taking into account (15), (9), and (10), we have

$$L_m(x, \{\psi\}, \lambda) = \int_a^b \left| \sum_{k=0}^{m} \lambda_k^m \psi_k(x) \psi_k(t) \right| dt$$
$$\leq \int_a^b \left| \sum_{j=0}^{p(i)} \Phi_j(x) \Phi_j(t) \right| dt + \int_a^b \left| \sum_{j=0}^{i-1} \sum_{k=m_j+1}^{m_{j+1}-1} \overline{\lambda}_k^m \overline{\chi}_{k-j}(x) \overline{\chi}_{k-j}(t) + \sum_{k=m_i+1}^{m} \overline{\lambda}_k^m \overline{\chi}_{k-i}(x) \overline{\chi}_{k-i}(t) \right| dt.$$

Since

$$\left| \overline{\lambda}_{m_{j-1}}^m - \overline{\lambda}_{m_{j+1}}^m \right| \leq \left| \Delta \overline{\lambda}_{m_{j-1}}^m \right| + \left| \Delta \overline{\lambda}_{m_j}^m \right|,$$

it is not difficult to see that the second integral represents the Lebesgue functions for some summation method regular in the sense of Toeplitz, and therefore, taking into account (8) and

(14), we find

$$\int_a^b \left| \sum_{i=0}^{i-1} \sum_{k=m_j+1}^{m_{j+1}-1} \bar{\lambda}_k^m \bar{\chi}_{k-j}(x) \bar{\chi}_{k-j}(t) + \sum_{k=m_i+1}^{m} \bar{\lambda}_k^m \bar{\chi}_{k-i}(x) \bar{\chi}_{k-i}(t) \right| dt = O(1).$$

Thus, by virtue of (11), we conclude that for $m \in [m_i, m_{i+1})$

$$L_m(x, \{\psi\}, \lambda) \leqslant L_{p(i)}(x, \{\Phi\}) + O(1) = O(\vartheta(p(i))).$$

Putting now $\gamma(m) = \vartheta(p(i))$ for $m \in [m_i, m_{i+1})$, we obtain estimate (4). We now define the coefficients of the series (5) as follows:

$$b_n = \begin{cases} a_n, & \text{for } n = m_i, \\ d_{n-i}, & \text{for } n \in (m_i, m_{i+1}), \end{cases}$$

where a_i $(i = 0, 1, ...)$ are the coefficients of the series (12), while $\{d_\nu\}$ is an arbitrary sequence of numbers satisfying the condition

$$\sum_{i=1}^{\infty} \left(\sum_{k=m_i+1}^{m_{i+1}-1} d_{k-i}^2 \right) f[\gamma(m_i)] < \infty.$$

Taking (13) into account, we find

$$\sum_{k=0}^{\infty} b_k^2 f[\gamma(k)] = \sum_{i=1}^{\infty} \sum_{k=m_i}^{m_{i+1}-1} b_k^2 f[\gamma(k)] =$$

$$= \sum_{i=1}^{\infty} a_i^2 f[\gamma(m_i)] + \sum_{i=1}^{\infty} \left(\sum_{k=m_i+1}^{m_{i+1}-1} d_{k-i}^2 \right) f[\gamma(m_i)] = O(1),$$

i.e. condition (6) is satisfied.

Let us now consider the λ-means of the series (5). Taking (9), (10) into account, for $m \in [m_i, m_{i+1})$ we obtain

$$U_m(x, \{\psi\}, \lambda) = \sum_{k=0}^{m} \lambda_k^m b_k \psi_k(x) =$$

$$= \sum_{j=0}^{p(i)} a_j \Phi_j(x) + \left\{ \sum_{j=0}^{i-1} \sum_{k=m_j+1}^{m_{j+1}-1} \lambda_k^m d_{k-j} \bar{\chi}_{k-j}(x) + \sum_{k=m_i+1}^{m} \lambda_k^m d_{k-i} \bar{\chi}_{k-i}(x) \right\}.$$

Since a series from L^2 with respect to the Haar system converges almost everywhere, it is also summable almost everywhere by any regular Toeplitz method. But the expression in braces is the mean of a certain regular Toeplitz summability method, and therefore the sequence $\{U_m(x, \{\psi\}, \lambda)\}$ diverges almost everywhere together with the sequence of partial sums

$$\left\{ \sum_{j=0}^{p(i)} a_j \Phi_j(x) \right\}$$

of the almost everywhere divergent series (12). Consequently, the series (5) is not summable almost everywhere by the method (λ), although condition (6) is satisfied. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.