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Abstract

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ON THE DECAY OF HARMONIC FUNCTIONS IN A CYLINDER

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The classical Phragmén–Lindelöf theorems for analytic functions can be divided into two types. In theorems of the first type, for various infinite domains one establishes the limiting growth under which there can exist a function regular in the given domain, bounded on the boundary and unbounded inside. In theorems of the second type, often called uniqueness theorems, one establishes how rapidly a nonidentically zero analytic function can decay uniformly inside the given domain.

In papers ([1−6]) analogues of theorems of the first type were established for harmonic functions of three variables. In the present note a uniqueness theorem is proved for harmonic functions of three variables in a half-cylinder, analogous to the classical uniqueness theorem for analytic functions decreasing in a half-plane.

Theorem 1. Let D be an arbitrary domain in the plane (x_1, x_2), and let $u(x, x_1, x_2)$ be a harmonic function of three variables in the half-cylinder $(x_1, x_2) \in D,\ 0 \leq x < \infty$. If, for arbitrarily small $a > 0,\ \eta > 0$, we have

$$|u(x, x_1, x_2)| < C \exp\left\{-a \exp \frac{\pi + \eta}{h} x\right\}, \tag{1}$$

where h is the diameter of the largest circle that can be placed in the domain D, then $u(x, x_1, x_2) \equiv 0$ in the whole half-cylinder.

Here and below C denotes a positive constant, not necessarily the same one.

For the proof we shall need the following well-known result. Let $f(z)$, $z = \sigma + i\tau$, be an analytic function, regular for $\sigma \geq 0$ and satisfying the conditions

$$f(z) = \exp\{o(|z|^2)\}, \quad |z| \to \infty,$$

$$|f(\sigma)| < C \exp\left(\frac{1}{\rho}\sigma \ln \sigma\right), \quad \rho > 0, \tag{2}$$

$$|f(i\tau)| < C\exp(-b|\tau|), \quad b > 0.$$

If, moreover, $b > \pi/2\rho$, then $f(z) \equiv 0$.

The assertions of Theorem 1 will follow from the lemmas proved below. Introduce the function

$$v(z, x_1, x_2) \equiv \int_0^\infty u(x, x_1, x_2) e^{xz}\, dx. \tag{3}$$

It follows from the conditions of Theorem 1 that $v(z, x_1, x_2)$, for $(x_1, x_2) \in D$, is an entire function of the variable z, satisfying the differential equation

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + z^2 v = u'_x(0, x_1, x_2) - zu(0, x_1, x_2). \tag{4}$$

Lemma 1. The function $v(z, x_1, x_2)$ can be represented in the form

$$v(z, x_1, x_2) = -\iint_G K(z, x_1-\rho_1, x_2-\rho_2)[u'_x(0, \rho_1, \rho_2) - zu(0, \rho_1, \rho_2)]\, d\rho_1\, d\rho_2 + v_1(z, x_1, x_2), \tag{5}$$

where

$$v_1(z, x_1, x_2) = \int_\gamma \left[K(z, x_1 - s_1, x_2 - s_2)\frac{\partial}{\partial n} v(z, s_1, s_2) - v(z, s_1, s_2)\frac{\partial}{\partial n} K(z, x_1 - s_1, x_2 - s_2)\right] dl; \tag{6}$$

$$K(z, \alpha_1, \alpha_2) = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int \frac{\exp(i\alpha_1\xi_1 + i\alpha_2\xi_2)}{\xi_1^2 + \xi_2^2 - z^2}\, d\xi_1\, d\xi_2; \tag{7}$$

G is the disk referred to in the statement of the theorem (for convenience, its center is taken to coincide with the origin); γ is the boundary of G; l is the length of the arc along γ, and $\partial/\partial n$ denotes differentiation in the direction of the exterior normal.

Proof. Consider the Fourier transform of the function $v(z, x_1, x_2)$

$$\tilde{v}(z, \xi_1, \xi_2) \equiv \iint_G v(z, x_1, x_2) \exp(-ix_1\xi_1 - ix_2\xi_2)\, dx_1\, dx_2.$$

From (4), by a straightforward calculation we obtain

$$\tilde{v}(z,\xi_1,\xi_2)=\frac{\psi(z,\xi_1,\xi_2)}{\xi_1^2+\xi_2^2-z^2};$$

where

$$\psi(z,\xi_1,\xi_2)=\iint_G [u'_x(0,x_1,x_2)-zu(0,x_1,x_2)]\exp(-ix_1\xi_1-ix_2\xi_2)\,dx_1\,dx_2+$$

$$+\int_\gamma\left[\exp(-ix_1\xi_1-ix_2\xi_2)\frac{\partial v}{\partial n}-v\,\frac{\partial}{\partial n}\exp(-ix_1\xi_1-ix_2\xi_2)\right]dl.$$

Applying the inverse Fourier transform and changing the order of integration, we obtain formula (5).

Lemma 2. On the real axis $z=\sigma$ we have

$$|v_1(\sigma,x_1,x_2)|<C\exp\left\{\frac{h}{\pi+\eta}\,\sigma\ln\sigma\right\},\qquad \sigma\to\infty.$$

Proof. From (6) and (7) it is clear that $v_1(z,x_1,x_2)$ grows on the real axis no faster than $v(z,x_1,x_2)$; therefore it suffices to estimate the function $v(\sigma,x_1,x_2)$. From (1) and (3) we obtain

$$|v(\sigma,x_1,x_2)|<C\int_0^\infty \exp\left\{-a\exp\frac{\pi+\eta}{h}x+\sigma x\right\}dx=C\Gamma\left(\frac{h\sigma}{\pi+\eta}\right),\qquad \sigma\to\infty.$$

The required estimate now follows with the aid of Stirling' s formula.

Lemma 3. For the kernel $K(z,\alpha_1,\alpha_2)$ and its normal derivative on the imaginary axis $z=i\tau$, we have

$$|K(i\tau,\alpha_1,\alpha_2)|<C|\tau|^{-1/2}e^{-r|\tau|},\qquad \frac{\partial}{\partial n}K(i\tau,\alpha_1,\alpha_2)\le|\tau|^{1/2}e^{-r|\tau|},\qquad (8)$$

where $r=\sqrt{\alpha_1^2+\alpha_2^2}$, and C depends on r.

Proof. From the easily verified equality

$$\iiint_{-\infty}^{+\infty}\frac{\exp(i\alpha_1\xi_1+i\alpha_2\xi_2+i\beta\tau)}{\xi_1^2+\xi_2^2+\tau^2}\,d\xi_1\,d\xi_2\,d\tau=\frac{2\pi^2}{\sqrt{\alpha_1^2+\alpha_2^2+\beta^2}}$$

it follows that $K(i\tau,\alpha_1,\alpha_2)$ can be represented in the form

$$K(i\tau, \alpha_1, \alpha_2) \equiv K(i\tau, r) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{\exp(i\beta\tau)}{\sqrt{r^2 + \beta^2}} \, d\beta. \tag{9}$$

If cuts are made from ir to $i\infty$ and from $-ir$ to $-i\infty$, then integration along the real axis in formula (9) may be replaced by integration along the shores of the upper cut for $\tau > 0$ and by integration along the shores of the lower cut for $\tau < 0$. We obtain

$$|K(i\tau, r)| = \frac{1}{2\pi} \int_r^{\infty} \frac{\exp(-s|\tau|)}{\sqrt{s^2 - r^2}} \, ds = \frac{1}{2\pi} \int_1^{\infty} \frac{\exp(-tr|\tau|)}{\sqrt{t^2 - 1}} \, dt.$$

Representing this integral as the sum of integrals over the intervals $(1, 2)$ and $(2, \infty)$, we obtain the first of estimates (8). To obtain the second estimate, we write

$$\left|\frac{\partial K}{\partial n}\right| = \left|\frac{\partial K}{\partial r}\right| \left|\frac{\partial r}{\partial n}\right| \leq \left|\frac{\partial K}{\partial r}\right|$$

and differentiate with respect to r under the integral sign.

Lemma 4. On the imaginary axis $z = i\tau$, for the function $v_1(z, x_1, x_2)$ we have

$$|v_1(i\tau, x_1, x_2)| < C|\tau|^{1/2} \exp\left\{-|\tau| \left(\frac{h}{2} - \sqrt{x_1^2 + x_2^2}\right)\right\}.$$

Proof. From the definition of $v(z, x_1, x_2)$ by formula (3) it follows that

$$|v(i\tau, x_1, x_2)| < C;$$

therefore it is enough, for $z = i\tau$, to estimate the double integral appearing on the right-hand side of equality (5). The required result follows immediately from formula (6) and estimates (8).

Proof of Theorem 1. We now subject x_1 and x_2 to the condition

$$\sqrt{x_1^2 + x_2^2} < \varepsilon < \frac{h}{2} \frac{\eta}{\pi + \eta}.$$

We shall have

$$\frac{h}{2} - \sqrt{x_1^2 + x_2^2} > \frac{\pi}{2} \frac{h}{\pi + \eta}.$$

Together with the results of Lemmas 2 and 4, this means that, for $x_1^2 + x_2^2 < \varepsilon^2$, $v_1(z, x_1, x_2)$ satisfies all the conditions of the auxiliary theorem formulated above (the fulfillment of the first of conditions (2) follows from the result of Lemma 2, since $v_1(z, x_1, x_2)$, obviously, grows no faster on the real axis). Therefore, for $x_1^2 + x_2^2 < \varepsilon^2$ we shall have $v_1(z, x_1, x_2) \equiv 0$, whence it follows that

$$v(z, x_1, x_2) =$$

$$= -\iint_G K(z, x_1 - p_1, x_2 - p_2)\,[u'_x(0, p_1, p_2) - zu(0, p_1, p_2)]\; dp_1\, dp_2.$$

Inverting the Laplace transform (3), we obtain from this

$$u(x, x_1, x_2) = \frac{1}{4\pi} \iint_G \left[\frac{\partial u}{\partial n}(0, p_1, p_2)\frac{1}{r} - u(0, p_1, p_2)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] dp_1\, dp_2. \qquad (10)$$

Here

$$r = \sqrt{x^2 + (x_1 - p_1)^2 + (x_2 - p_2)^2}, \qquad \frac{\partial}{\partial n} = -\frac{\partial}{\partial x}$$

and $x_1^2 + x_2^2 < \varepsilon^2$. By the uniqueness theorem for harmonic functions, equality (10) will be valid in the entire cylinder $(x_1, x_2) \in G,\ 0 \le x < \infty$.

We pass to the final stage of the proof. Denote by S the lateral surface of the cylinder $(x_1, x_2) \in G,\ 0 \le x < \infty$. Then, according to

by the well-known formula, for any point $A(x, x_1, x_2)$ we have:

$$\frac{1}{4\pi} \iint_{G+S} \left(\frac{\partial u}{\partial n}\frac{1}{r} - \frac{\partial}{\partial n}\left(\frac{1}{r}\right) u\right) d\sigma = \begin{cases} u(A) & \text{(the point } A \text{ is inside the cylinder)}, \\ 0 & \text{(the point } A \text{ is outside the cylinder)}. \end{cases} \qquad (11)$$

Comparing (10) and (11), we find that

$$\iint_S \left(\frac{\partial u}{\partial n}\frac{1}{r} - \frac{\partial}{\partial n}\left(\frac{1}{r}\right) u\right) d\sigma = 0,$$

if the point A is inside the cylinder. But this integral represents a function harmonic everywhere outside S, and therefore it is identically equal to zero. Now it follows from (11) that the integral on the right-hand side of (10) is equal to zero for all points external to the cylinder, and consequently it is identically equal to zero. This observation proves our theorem.

Let us note that the constant π/h found is exact for many domains D (for example, for a rectangle, a disk, a semicircle). Namely, following ([3]), we shall call the width of the smallest strip containing D the outer width $H = H(D)$ of the domain D. The diameter of the largest disk $h = h(D)$ that can be placed in the domain D will be called the inner width of the domain D.

Theorem 2. For domains with equal outer and inner width, the constant $\pi/h(D)$ is sharp.

Proof. Without loss of generality, one may suppose that the smallest strip containing the domain D is bounded by the lines $x_1 = \pm H/2$. Consider the harmonic function

$$u(x, x_1, x_2) = \operatorname{Re}\left\{\exp\left(-\exp\frac{\pi-\varepsilon}{H}(x+ix_1)\right)\right\}.$$

We have

$$|u(x, x_1, x_2)| = \exp\left\{-\cos\frac{\pi-\varepsilon}{H}x_1 \cdot \exp\frac{\pi-\varepsilon}{H}x\right\}.$$

For $|x_1| < H/2$ and any $\varepsilon > 0$ we shall have

$$\cos\frac{\pi-\varepsilon}{H}x_1 > \cos\frac{\pi-\varepsilon}{2} = a_\varepsilon,$$

i.e.

$$|u(x, x_1, x_2)| < \exp\left\{-a_\varepsilon \exp\frac{\pi-\varepsilon}{H}x\right\}.$$

For domains where $H(D) \neq h(D)$ (for example, for a triangle), the question of the exact constant remains unresolved.

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REFERENCES

1. M. A. Evgrafov, I. A. Chegis, DAN, **134**, No. 2 (1960).
2. I. A. Chegis, DAN, **134**, No. 3 (1960).
3. I. S. Arshon, M. A. Evgrafov, DAN, **142**, No. 4 (1962).
4. I. S. Arshon, M. A. Evgrafov, DAN, **143**, No. 1 (1962).

5. M. A. Evgrafov, I. S. Arshon, DAN, **147**, No. 4 (1962).

6. Sh. Yarmukhamedov, DAN, **152**, No. 3 (1963).

Note: Figure translations are in progress. See original paper for figures.