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Abstract

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ON A CONSTRUCTIVE PRINCIPLE OF DUALITY IN THE CLASS OF CONTINUOUS FUNCTIONS DECREASING MONOTONICALLY TO ZERO AND CONVEX ON THE SEMI-AXIS

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Let $\omega(t)$ be a modulus of continuity defined on the semi-axis $0 \leq t < \infty$, i.e., a continuous, nondecreasing, semiadditive function for which $\omega(0) = 0$ (see ([1]), Ch. III). For any nonnegative value of the constant M consider the class MH_ω of all functions $g(x)$ defined on the semi-axis $0 \leq x < \infty$ and satisfying the condition

$$|g(x_1) - g(x_2)| \leq M\omega(|x_1 - x_2|). \tag{1}$$

Denote by $E_M^\omega(f)$ the best uniform approximation on $[0, \infty)$ of a bounded function $f(x)$ by functions $g(x)$ satisfying condition (1), i.e., the quantity

$$E_M^\omega(f) = \inf_{g \in MH_\omega} \sup_{0 \leq x < \infty} |f(x) - g(x)|. \tag{2}$$

It is obvious that $E_M^\omega(f)$, as a function of M, defined on the positive semi-axis $0 \leq M < \infty$, is always nonincreasing.

Assuming the modulus of continuity $\omega(t)$ to be convex and unbounded as $t \to \infty$, in this note we shall give an exhaustive characterization of the best approximation $E_M^\omega(f)$ in the class of bounded monotone functions $f(x)$, convex (or concave) on the semi-axis.

The following theorem of the type of the well-known duality theorems holds.

Theorem. *For any convex (upward) and unbounded on $[0, \infty)$ modulus of continuity $\omega(t)$, the class of all continuous functions $f(x)$, monotonically decreasing to zero and convex (downward) on the semi-axis $0 \leq x < \infty$, coincides with the class of their best approximations $E(M) = E_M^\omega(f)$.*

This theorem, showing that for any convex and unbounded modulus of continuity $\omega(t)$ a nonnegative and nonincreasing function $E(M)$ on the semi-axis

$0 \leq M < \infty$ is the best uniform approximation $E_M^\omega(f)$ of some bounded monotone and convex on $[0, \infty)$ function $f(x)$ if and only if it is continuous and convex, in particular, holds for $\omega(t) = t$, i.e., when considering the best uniform approximation $E'_M(f)$ by absolutely continuous functions whose derivative is almost everywhere bounded by one or another number $M \geq 0$.

It should be noted that the requirement of unboundedness of the modulus of continuity $\omega(t)$, which plays an essential role in the proof, is necessary in order that the duality principle formulated in the theorem hold.

One can indicate a continuous function $E(M)$, monotonically decreasing to zero and convex on the semi-axis $0 \leq M < \infty$, which for no bounded function $f(x)$, and for no modulus of continuity $\omega(t)$ bounded on $[0, \infty)$, will be the best approximation $E_M^\omega(f)$. It suffices

take, for example, the function

$$E(M) = \begin{cases} 1 - \sqrt{M}, & 0 \leq M \leq 1, \\ 0, & M \geq 1. \end{cases}$$

It is easy to see that if $\omega(t)$ is bounded as $t \to \infty$, then, whatever bounded function $f(x)$ we take, for sufficiently small values of M the inequality

$$E_M^\omega(f) \geq E_0^\omega(f) - \frac{1}{2} M \sup_{t>0} \omega(t)$$

will hold.

Hence it is clear that either $E_0^\omega(f) \neq 1$, or, for all sufficiently small positive M, one will have $E_M^\omega(f) > E(M)$.

The proof of the fact that, if a nonnegative continuous and nonincreasing on $[0, \infty)$ function $f(x)$ is convex, then its best approximation $E_M^\omega(f)$ is a continuous convex function of M, tending to zero as $M \to \infty$, under the condition $f(0) = 0$, is connected with consideration of the function

$$\Omega(x) = \min\{f(x),\, M\omega(x)\} \tag{3}$$

and of the properties of the difference $f(x) - \Omega(x)$*.

To prove the converse assertion, namely that every continuous function $f(M)$ monotonically decreasing to zero is the best uniform approximation $E_M^\omega(\varphi)$ of some other function $\varphi(x)$ of the same kind, one establishes the existence of a sequence of monotone and convex functions $\varphi_n(x)$ $(n = 0, 1, 2, \ldots)$ possessing the property that, for any $q = 0, 1, \ldots, 2^n \cdot n$,

$$E_{q/2^n}^\omega(\varphi_n) = f\left(\frac{q}{2^n}\right). \tag{4}$$

A subsequent application of Helly' s selection principle, taking into account the special properties of the functions $\varphi_n(x)$, leads to a function $\varphi(x)$ for which $E_M^\omega(\varphi) = f(M)$ for every $M \geq 0$.

The theorem stated remains valid if, instead of the class of all functions $g(x)$ satisfying condition (1), one considers that part of it which contains only functions convex on $[0, \infty)$.

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REFERENCES

1. A. F. Timan, *Theory of approximation of functions of a real variable*, Moscow, 1960.

2. A. F. Timan, DAN, 140, No. 2, 307 (1961).

* In this connection see (1).

Note: Figure translations are in progress. See original paper for figures.