MATHEMATICS

1963

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Abstract

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MATHEMATICS

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A GENERALIZATION OF A THEOREM OF A. F. TIMAN

(Presented by Academician S. N. Bernstein on 20 VII 1962)

1°. For approximations of functions, defined on a finite interval, by algebraic polynomials, the following theorem of A. F. Timan is known ((5), see also (6, § 5.2), which is a strengthening of the classical Jackson theorem:

If $f(x)$ has on $[-1,1]$ a continuous derivative of order r, then for every integer $n > r$ there exists an algebraic polynomial $P_n(x)$ of degree $\le n$ such that

$$|f(x) - P_n(x)| \le A_r \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^r \omega_1\left(f^{(r)}; \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right), \quad (1)$$

$$-1 \le x \le 1;$$

A_r depends only on r.

This theorem, connected with the presence of corner points on the interval, was developed in a number of works in various directions. V. K. Dzyadyk (4) and G. Freud (9) obtained an analogous result for classes of functions with a prescribed second modulus of smoothness of the r-th derivative. In work (3) result (1) was generalized to the case of approximation on a finite system of intervals. Analogous features arising in passing to approximation by entire functions on one or two infinite intervals were discovered in papers by the author (2) and by R. M. Trigub (7).

In the present work all the results indicated above on approximation by algebraic polynomials are generalized to classes of functions with arbitrarily prescribed differential-difference properties.

Let, for a function $f(x)$ defined on $[a,b]$,

$$\omega_r(f;t) = \sup_{a \le x \le b} \sup_{|h| \le t} |\Delta_h^r f(x)|$$

be the r-th modulus of smoothness, and

$$E_n(f;a,b) = \inf_{c_k} \sup_{a\le x\le b} \left| f(x) - \sum_{k=0}^{n} c_{kx}^{k} \right|.$$

Theorem 1. *For every bounded function $f(x)$, defined on $[-1,1]$, for any natural r and $n \ge r-1$, there exists an algebraic polynomial $P_n(x)$ of degree $\le n$ such that*

$$|f(x) - P_n(x)| \le A_r \omega_r \left(f; \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} \right), \tag{2}$$

where A_r is a constant depending only on r.

Remark 1. If $f(x)$ has a bounded k-th derivative, then

$$\omega_r(f;t) \le t^k \omega_{r-k}(f^{(k)};t).$$

Therefore Timan' s and Dzyadyk—Freud' s theorems follow directly from inequality (2).

Theorem 2. *Let a bounded function $f(x)$ be defined on the set*

$$K = \bigcup_{j=1}^{p} [a_j, b_j] \quad \text{and} \quad \lambda(x) = \frac{2\sqrt{(b_j - x)(x - a_j)}}{b_j - a_j} \quad \text{for } x \in [a_j, b_j]\ (j = 1, 2, \dots, p).$$

Then, for any natural numbers r and $n \ge r-1$, there exists an algebraic polynomial $P_n(x)$ of degree $\le n$ such that

$$|f(x) - P_n(x)| \le A_r(K)\, \omega_r \left(f; \frac{\lambda(x)}{n} + \frac{1}{n^2} \right), \tag{3}$$

where $A_r(K)$ depends only on r and K.

Remark 2. By virtue of Remark 1, Theorem 2 implies the author' s result noted above (3).

Theorem 3. Let $f(x)$ be a bounded function defined on $[0,1]$. Then, for any natural numbers r and $n \ge r-1$,

$$E_n(f;0,1) \le B_r \omega_r \left(f; \frac{1}{n+1} \right), \tag{4}$$

where B_r depends only on r.

Remark 3. Theorem 3 does not follow from Theorem 1, since $B_r = o(A_r)$ as $r \to \infty$.

The following theorem shows that functions whose differential-difference properties deteriorate near the endpoints of the interval can be approximated more accurately than in Theorem 3.

Theorem 4. Suppose that, under the conditions of Theorem 3, for some natural number r,

$$|\Delta_h^{2r} f(x)| \leq \varphi_1[h^2\psi(x)] \qquad \text{for } 0 < x;\ x + 2rh \leq 1,$$

$$|\Delta_h^{2r} f(x)| \leq \varphi_2(\delta) \qquad \text{for } 0 \leq x \leq \delta,\ |h| \leq \delta.$$

Here φ_1 and φ_2 are monotonically increasing to $+\infty$ functions, while $\psi(x) \geq 0$ and tends to ∞ as $x \to +0$. Denote by t_n the (unique) root of the equation

$$\varphi_1[n^{-2}\psi(t^2)] = \varphi_2(t^2).$$

Then, for $n \geq 2r - 1$, we have

$$E_n(f; 0, 1) \leq C_r \varphi_2(t_n^2). \tag{5}$$

To illustrate Theorem 4, let us give the following example. Let $f(x) = x^p \sin x^{-q}$, $p > 0$, $q \geq 0$. It can be shown that (4) gives $E_n(f; 0, 1) \leq A_{p,q} n^{-p/(q+1)}$, whereas the application of (5) gives $E_n(f; 0, 1) \leq C_{p,q} n^{-2p/(2q+1)}$. The latter inequality is order-sharp.

2°. Let us outline the proofs of the formulated theorems. Let $T_n = \{I_\nu\}_{\nu=1}^{n+1}$ ($R_n = \{I_\nu\}_{\nu=1}^{n+1}$) be the Chebyshev (uniform) partition of the interval, i.e.

$$I_\nu = [\cos\theta_\nu, \cos\theta_{\nu+1}] \qquad \left(= \left[\frac{\nu-1}{n}, \frac{\nu}{n}\right]\right),$$

where $\theta_\nu = \dfrac{n-\nu+1}{n}\pi$. Let

$$I_{\nu,r} = \bigcup_{s=0}^{r-1} I_{\nu+s}.$$

Theorems 1, 3, and 4 are obtained from the following assertion (and Whitney' s theorem (8)):

Theorem 5. For every function $f(x)$ bounded on $[-1, 1]$ and for every $n \geq r-1$, there exists an algebraic polynomial $P_N(f; x)$ of degree $N \leq 3(n+1)r$ such that

$$|f(x) - P_N(f;x)| \le A_r \sum_{\nu=1}^{n-r+2} E_{r-1}(f; I_{\nu,r})\, \Phi_{r+2}^{(n)}(t-\theta_\nu). \tag{6}$$

Here $\Phi_s^{(n)}(x) = (1 + n|x|)^{-s}$, $t = \arccos x$. The constant A_r for Chebyshev (uniform) approximation is

$$\le A_0 \left(\frac{r\pi}{e}\right)^{2r+1} \qquad (\le A_0 \pi^{4r}).$$

For the proof we shall need some auxiliary results. Let $S = \{[x_\nu, x_{\nu+1}]\}_{\nu=1}^{n+1}$ be an arbitrary partition of the interval $[-1,1]$; $L[a_1, \dots, a_{r+1}](f)$ is the Lagrange interpolation polynomial, po-

constructed for the function $f(x)$ at the points $a_1, \dots, a_{r+1}$, and $f(a_1, \dots, a_{r+1})$ is the r-th divided difference of $f(x)$ at the same points. Let

$$L_S^{(r)}(f;x) \overset{\text{def}}{=} \begin{cases} L[x_\nu, \dots, x_{\nu+r-1}](f), & \text{for } x \in [x_\nu, x_{\nu+1}],\ \nu = 1, \dots, n-r+2, \\ L[x_{n-r+2}, \dots, x_{n+1}](f), & \text{for } x \in [x_{n-r+2}, x_{n+1}]. \end{cases}$$

Then the following is true.

Lemma 1. *The identity holds*

$$L_S^{(r)}(f;x) = P_{r-1}(x) + \sum_{\nu=1}^{n-r+1} a_{\nu+1}(f) l_{\nu+1}(x), \qquad -1 \le x \le 1, \tag{7}$$

where $P_{r-1}(x) = L[x_1, \dots, x_r](f)$, $a_{\nu+1}(f) = (x_{\nu+1} - x_\nu) f(x_\nu, \dots, x_{\nu+r})$,

$$l_{\nu+1}(x) = \frac{1}{2}\{1 + \operatorname{sgn}[x_{\nu+1} - x]\} \prod_{s=1}^{r-1} (x - x_{\nu+s}).$$

We shall also need the following.

Lemma 2. Let

$$\chi(\theta;t) = \frac{1}{2}\{1 + \operatorname{sgn}[\cos\theta - \cos t]\}.$$

Then for every natural number n there exists a trigonometric polynomial $K_N(\theta, t)$ of degree $N \le 2mn$ such that

$$|\chi(\theta;t) - K_N(\theta;t)| \le \pi^{2m-1} \Phi_{2m-1}^{(n)}(\theta - t). \tag{8}$$

We proceed to the proof of Theorem 5. Put $S = R_n (= T_n)$ in Lemma 1. Then we have

$$|f(x) - L_S^{(r)}(f;x)| \le A_1(r) E_{r-1}(f; I_{\nu(x),r}), \tag{9}$$

where $\nu(x)$ is such that $x \in I_{\nu(x)}$.

Next, by the definition of $l_\nu(x)$, we have

$$l_{\nu+1}(\cos t) = \chi(\theta_{\nu+1}; t) \prod_{s=1}^{r-1} (\cos t - \cos\theta_{\nu+s}).$$

Therefore, with the aid of Lemma 2 we obtain

$$|l_\nu(\cos t) - K_{N_1,\nu}(t)| \le A_2(r) \frac{\sin^{r-1}\theta_{\nu+1}}{n^{r-1}} \Phi_{r+2}^{(n)}(\theta_\nu - t),$$

where

$$K_{N_1,\nu+1}(t) = K_N(\theta_{\nu+1}; t) \prod_{s=1}^{r-1} (\cos t - \cos\theta_{\nu+s})$$

is an even algebraic polynomial of degree $N_1 \le 3(n+1)r$. We now consider the even trigonometric polynomial

$$K_{N_1}(f;t) = P_{r-1}(\cos t) + \sum_{\nu=1}^{n-r+1} a_{\nu+1}(f) K_{N_1,\nu+1}(t).$$

Taking into account that

$$|a_{\nu+1}(f)| \le A_3(r) \frac{n^{r-1}}{\sin^{r-1}\theta_{\nu+1}} E_{r-1}(f; I_{\nu,r}),$$

we obtain

$$|L_S^{(r)}(f;\cos t) - K_{N_1}(f;t)| \le A_4(r) \sum_{\nu=1}^{n-r+1} E_{r-1}(f; I_{\nu,r}) \Phi_{r+2}^{(n)}(\theta_\nu - t). \tag{10}$$

The theorem follows from (9) and (10).

Theorem 2 is obtained from Theorem 1 and the following lemma.

Lemma 3. Let $I_s = [a_s, b_s]$, $s = 1, 2, 3$, be nonintersecting intervals, and let $Q_n(x)$ be a polynomial of degree n such that $|Q_n(x)| \leq M$ for $x \in I_2$ (I_2 is the middle interval). Then there exists a polynomial $R(x)$ of degree $4sn$ such that

$$|Q_n(x) - R(x)| \leq AM\rho^n \quad \text{for } x \in I_2,$$

$$|R(x)| \leq AM\rho^n \quad \text{for } x \in I_1 \cup I_3.$$

Here $A > 0$, the natural number s, and $0 < \rho < 1$ depend only on a_s, b_s, $s = 1, 2, 3$.

Proof. By S. N. Bernstein' s inequality (see [1], p. 21) we have $|Q_n(x)| < Ml^n$ for $x \in I_1 \cup I_3$, where $l > 1$ depends only on a_s, b_s, $s = 1, 2, 3$. Moreover, it is not difficult to construct a polynomial $S(x)$ of degree $4n$ such that $|1 - S(x)| < Ar^n$ for $x \in I_2$ and $|S(x)| < Ar^n$ for $x \in I_1 \cup I_3$, where $A > 0$ and $0 < r < 1$ depend only on a_s, b_s ($s = 1, 2, 3$).

For example, one may put

$$S(x) = P\left(\frac{2x - b_1 - a_2}{2M - b_1 - a_2}\right) P\left(\frac{b_2 + a_3 - 2x}{2N - b_2 - a_3}\right),$$

where

$$M = \max\{2b_3 - b_1 - a_2,\ b_1 + a_2 - 2a_1\},$$

$$N = \{2b_3 - a_3 - b_2,\ b_2 + a_3 - 2a_1\},$$

$$P(x) = g\int_0^1 [1 - (x - t)^2]^n\, dt,$$

and g is such that $P(1) = 1$. If s is the least integer such that $lr^s < 1$, then the required polynomial is

$$R(x) = Q_n(x)[S(x)]^s.$$

Received
12 VII 1962

CITED LITERATURE

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