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Abstract

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ON THE SPECTRUM OF A NON-SELF-ADJOINT DIFFERENTIAL OPERATOR OF EVEN ORDER

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In the work of M. A. Naimark (^{1}) it was proved that the differential operator $-y'' + q(x)y$, considered on the half-line ($0 \le x < \infty$), in the case where $q(x)$ is a complex-valued function satisfying the condition $|q(x)| < e^{-\varepsilon_0 x}$, has no positive eigenvalues. This fact, however, does not generalize to the case of differential operators of order $2n$ ($n \ge 2$).

Consider in $\mathcal{L}_2(-\infty, \infty)$ the differential operator

$$Ly \equiv (-1)^n \frac{d^{2n}y}{dx^{2n}} + q(x)y, \tag{1}$$

where $q(x)$ is, generally speaking, a complex-valued function satisfying, for some fixed $\varepsilon_0 > 0$, the condition

$$|q(x)| \le Ce^{-\varepsilon_0|x|} \qquad (-\infty < x < \infty). \tag{2}$$

The present note is devoted to the proof of the following theorem:

Theorem. *The eigenvalues of the operator L, under condition (2), can form only a finite set.*

Moreover, even in the case where $q(x) \equiv 0$ outside some finite interval $[a, b]$, the operator L for $n \ge 2$ may have positive eigenvalues. In the complex λ-plane we shall indicate the radius of a circle, depending on $q(x)$, outside which the operator certainly has no eigenvalues.

We pass to the proof. For definiteness we shall assume n to be an even number.

Let us find the resolvent of the operator $d^{2n}y/dx^{2n}$ in $\mathcal{L}_2(-\infty, \infty)$. To this end one must solve the equation

$$\frac{d^{2n}y}{dx^{2n}} = s^{2n}y + \delta(x), \qquad 0 < \arg s < \frac{\pi}{n}.$$

With the aid of the Fourier transform and the theory of residues we find the resolvent kernel

$$K_s(x-y) = \frac{\pi i}{ns^{2n-1}} \sum_{k=0}^{n-1} \varepsilon_k e^{is\varepsilon_k|x-y|};$$

here by ε_k are denoted those roots of degree $2n$ of 1 for which $\operatorname{Im} s\varepsilon_k > 0$.

Therefore the resolvent of the unperturbed operator $d^{2n}y/dx^{2n}$ is the integral operator

$$R_s^0 f(x) = \frac{\pi i}{ns^{2n-1}} \int_{-\infty}^{\infty} K_s(x,y) f(y)\,dy, \qquad f \in \mathcal{L}_2. \tag{3}$$

Let us now consider the equation for determining the resolvent of the operator L

$$\frac{d^{2n}y}{dx^{2n}} + q(x)y - s^{2n}y = f, \qquad f \in \mathcal{L}_2(-\infty,\infty).$$

Applying to this equality the resolvent R_s^0 and denoting by T_s the operator whose action on $y(x) \in \mathcal{L}_2(-\infty,\infty)$ consists in first multiplying $y(x)$ by $q(x)$, and then applying R_s^0 to $q(x)y(x) \in \mathcal{L}_2(-\infty,\infty)$, by virtue of condition (2), we obtain the integral equation

$$y(x) + T_s y(x) = R_s^0 f; \tag{4}$$

for $0 < \arg s < \pi/n$, by virtue of $\operatorname{Im} s\varepsilon_k > 0$, the operator T_s will be a Hilbert-Schmidt operator, and it depends analytically on s; the kernel of this operator is a continuous and bounded function, and therefore Fredholm theory is applicable. The solution of (4) can be written with the aid of Fredholm determinants

$$y_s(x) = R_s^0 f(x) + \int_{-\infty}^{\infty} \frac{D(x,y,s)}{D(s)} R_s^0 f(y)\,dy,$$

where

$$D(s) = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{d_k(s)}{k!}, \tag{5}$$

where

$$d_k(s) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \begin{vmatrix} K_s(\alpha_1, \alpha_1) & \dots & K_s(\alpha_1, \alpha_k) \\ K_s(\alpha_2, \alpha_1) & \dots & K_s(\alpha_2, \alpha_k) \\ \cdot & \cdot & \cdot & \cdot \\ K_s(\alpha_k, \alpha_1) & \dots & K_s(\alpha_k, \alpha_k) \end{vmatrix} \cdot \prod_{j=1}^{k} q(\alpha_j)\, d\alpha_j,$$

$$K_s(x, y) = \frac{\pi i}{n s^{2n-1}} \sum_{k=0}^{n-1} \varepsilon_k e^{is\varepsilon_k |x-y|}, \qquad 0 < \arg s < \frac{\pi}{n}. \tag{6}$$

Lemma 1. *$D(s)$ admits analytic continuation through the positive real semi-axis into the domain S, which is defined by the inequalities*

$$|s| \geq s_0 > 0, \qquad \operatorname{Im} s > -\varepsilon_0/2, \qquad \operatorname{Re} s > 0, \qquad -\pi/n < \arg s < \pi/n.$$

In the domain S, $K_s(x, y)$ is an analytic function of s for fixed x and y, and admits the estimate

$$|K_s(x, y)| \leq \frac{\pi}{|s|^{2n-1}} e^{\frac{\varepsilon_0}{2}|x-y|}.$$

This estimate can be strengthened by the triangle inequality:

$$|K_s(x, y)| \leq \frac{\pi}{|s_0|^{2n-1}} e^{\frac{\varepsilon_0}{2}(|x|+|y|)}. \tag{7}$$

Using condition (2) and Hadamard’ s lemma, we obtain that in the domain S

$$|d_k(s)| \leq \left(\frac{\pi B!}{n s_0^{2n-1}} \sqrt{k} \right)^k, \tag{8}$$

where

$$B = \int_{-\infty}^{\infty} |q(x)| e^{\varepsilon_0 |x|}\, dx.$$

By virtue of estimate (8), by d’ Alembert’ s test, the functional series defining $D(s)$ and consisting of analytic functions converges uniformly in S. Lemma 1 is proved.

It follows from Lemma 1 that in the domain $0 \leq \arg s < \pi/n$, $s \neq 0$, the operator L can have at most a countable set of eigenvalues, with possible accumulation points at infinity and at $s = 0$.

Choose the number L_0 from the condition

$$1 = \sum_{k=1}^{\infty} \frac{k^{k/2} L_0^k}{k!},$$

and choose $s_0 > 0$ so that

$$q = \frac{\pi B}{n s_0^{2n-1}} < L_0. \tag{9}$$

Then, for $|s| \geq s_0$ and $s \in S$,

$$|D(s)| \geq 1 - \left| \sum_{k=1}^{\infty} \frac{k^{k/2}}{k!} q^k \right| \neq 0,$$

i.e., for sufficiently large $|s|$, $D(s)$ has no zeros.

Corollary. *The eigenvalues of the operator L form a bounded set, the only limiting point of which can be the point $\lambda = 0$.*

At the same time, we can also give an effective estimate of the disk in the complex plane $\lambda = s^{2n}$ outside which the operator L certainly has no eigenvalues. Namely, from formula (9) it follows immediately that, for $|\lambda| \geq R$, where

$$R = \left(\frac{\pi \int_{-\infty}^{\infty} |q(x)| e^{\varepsilon_0 |x|} \, dx}{n L_0} \right)^{\frac{2n}{2n-1}}, \tag{10}$$

the operator L has no eigenvalues.

It remains to show that $s = 0$ is not an accumulation point of the eigenvalues of the operator L.

Consider

$$\frac{d^{2n} y}{dx^{2n}} + q(x) y = s^{2n} y \quad (-\infty < x < \infty), \qquad 0 \leq \arg s < \frac{\pi}{n}, \tag{11}$$

where $q(x)$ satisfies condition (2).

Lemma 2. *The differential equation (11) has $2n$ linearly independent solutions which have the asymptotics*

$$y_i(x, s) = e^{\rho_i s x} \left(1 + o(e^{\varepsilon_0 x})\right) \quad (x \to -\infty), \tag{12}$$

and, for each fixed x, $y_i(x,s)$ is an analytic function of s in the disk $2|s| < \varepsilon_0$; here ρ_i are the roots of degree $2n$ of 1 $(i = 1, 2, \dots, 2n)$.

We outline the proof. From the differential equation (11), for each fixed ρ_i we pass to the integral equation

$$y_i(x,s) = e^{\rho_i x s} - \int_{-\infty}^{x} K(x,z,s) y_i(z,s)\, dz, \tag{13}$$

where

$$K(x,z,s) = \frac{\sum_{i=1}^{2n} \rho_i e^{\rho_i s(x-z)} q(z)}{2n\, s^{2n-1}}.$$

By expanding in a Taylor series, we verify that $K(x,z,s)$ is a function analytic in s over the whole plane for fixed x and z, and ⋯

the estimate holds

$$|K(x,z,s)| \le \frac{|x-z|^{2n-1}}{(2n-1)!} e^{2|s||x-z| - \varepsilon_0 |z|}. \tag{14}$$

We first prove the existence of the required solutions for the integral equations, using estimate (14) by the method of successive approximations.

Lemma 3. *The differential equation (11) has $2n$ linearly independent solutions that have the asymptotics*

$$u_i(x,s) = e^{\rho_i s x} \left(1 + o\left(e^{-\varepsilon_0 x}\right)\right) \quad (x \to +\infty), \qquad i = 1, 2, \dots, 2n, \tag{15}$$

and $u_i(x,s)$, for each fixed x, are analytic functions of s in the disk $2|s| < \varepsilon_0$, for each fixed $x \in (-\infty, \infty)$.

The proof is analogous to the proof of Lemma 2; one need only consider the integral equation

$$u_i(x,s) = e^{\rho_i s x} + \int_{x}^{\infty} K(x,z,s) u_i(z,s)\, dz.$$

Lemma 4. *The point $\lambda = 0$ is not an accumulation point of the eigenvalues of the operator L.*

The proof follows from Lemmas 2 and 3, and also from the fact that the determinant $D(s) \neq 0$ in the domain S, $s \neq 0$.

In conclusion, I take this opportunity to express my gratitude to Prof. M. A. Naimark and R. S. Ismagilov for valuable advice and discussion of the results obtained.

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References

[1] M. A. Naimark, *Tr. Mosk. matem. obshch.*, **3**, 181 (1954).

Note: Figure translations are in progress. See original paper for figures.

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