F. A. BEREZIN

1963

SovietRxiv

Abstract

Full Text

F. A. BEREZIN

ON CANONICAL TRANSFORMATIONS IN THE SECOND-QUANTIZATION REPRESENTATION

(Presented by Academician I. G. Petrovskii, January 10, 1963)

1.

Consider a Hilbert space with an involution L. Let $\hat{a}(f)$, $\hat{a}^*(f)$, $f \in L$, be linear functionals on L with values in the set of linear operators in the Hilbert space $\mathscr{H}$. We shall assume that the operators $\hat{a}^*(f)$, $\hat{a}(f)$ satisfy the usual (fermion or Bose) commutation relations, that they form an irreducible family, and that in the space $\mathscr{H}$ there exists a vacuum vector $\hat{\Phi}_0$: $\hat{a}(f)\hat{\Phi}_0 = 0$.

Let Φ, Ψ be operators in L having a common everywhere dense domain of definition D, endowed with its own topology and invariant with respect to the involution. We denote by $\widetilde{L}$ the space of continuous linear functionals on D.

We shall assume that $\widetilde{L}$ contains L as a dense set. The value of an element $F \in \widetilde{L}$ on $\varphi \in D$ will be denoted in the same way as the scalar product in L: $F(\varphi) = (F, \varphi^*)$ ($*$ is the involution in L). Define the operators $\overline{\Phi}, \overline{\Psi}$: $f\overline{\Phi} = (f^*\Phi)^*$, $f\overline{\Psi} = (f^*\Psi)^*$, $f \in L$.

Consider the operator linear functionals in D:

$$\hat{b}(f) = \hat{a}(f\Phi) + \hat{a}^*(f\Psi) + (F, f^*), \qquad \hat{b}^*(f) = \hat{a}(f\overline{\Psi}) + \hat{a}^*(f\overline{\Phi}) + (f, F). \quad (1)$$

If transformation (1) is invertible and the operators $\hat{b}(f)$, $\hat{b}^*(f)$ satisfy the same permutation relations as the operators $\hat{a}(f)$, $\hat{a}^*(f)$, then it is called a **linear canonical transformation**. In the fermion case $F = 0$.

It is well known that if the space L is finite-dimensional, then in the space $\mathscr{H}$ there exists a unitary operator $\widehat{U}$ which generates transformation (1):

$$\hat{b}(f) = \hat{U}\hat{a}(f)\hat{U}^{-1}, \qquad \hat{b}^*(f) = \hat{U}\hat{a}^*(f)\hat{U}^{-1}. \quad (2)$$

In the general case this is not always so. We shall agree to call a canonical transformation **proper** if there exists a unitary operator satisfying condition (2), and **improper** in the opposite case. It is known (), *that in order that the* *canonical transformation (1) be proper, it is necessary and sufficient that the operator Ψ be a Hilbert–Schmidt operator and that the functional F belong to L.*

We note that in order that transformation (1) be canonical, definite relations must hold between the operators Φ and Ψ. These relations imply that, in the case when the transformation is proper, the operator Φ is bounded**. Thus, in this case the domain D on which the operator functionals $\hat{b}(f)$ and $\hat{b}^*(f)$ are defined coincides with L.

Let $\mathcal{A}$ be some canonical transformation given by operators Φ, Ψ and a functional F, defined on the domain D. Consider a sequence of canonical transformations $\mathcal{A}_n$ having the following properties: 1) all operators Φ_n, Ψ_n and functionals F_n are defined on the domain D; 2) as $n \to \infty$, $\|\Phi_n f - \Phi f\| \to 0$, $\|\Psi_n f - \Psi f\| \to 0$, where f is an arbitrary element of D; $F_n \to F$ in the strong topology of the space $\widetilde{L}$.

In this case we shall call the transformation $\mathcal{A}$ the **limit of the transformations $\mathcal{A}_n$**.

---

* In (*) this theorem is proved for homogeneous transformations; however, by the same method a general result can be obtained.

** In the fermion case the operators Φ and Ψ are bounded for any canonical transformation.

Theorem 1. *Every linear canonical transformation is the limit of proper linear canonical transformations.*

2. Let $\mathcal{A}_n$ be a sequence of proper canonical transformations converging to an improper transformation $\mathcal{A}$; let $\hat{U}_n$ be the unitary operators in $\mathcal{H}$ implementing the transformations $\mathcal{A}_n$.

Definition. An operator $\hat{A}$ in $\mathcal{H}$, defined on the domain $D_{\hat{A}}$, sustains the improper canonical transformation $\mathcal{A}$ if, for every n, the operators $\hat{U}_n \hat{A} \hat{U}_n^{-1}$ are defined on $D_{\hat{A}}$, and for every $f \in D_{\hat{A}}$ there exists, in the strong sense, the limit $\lim_{n\to\infty} \hat{U}_n \hat{A} \hat{U}_n^{-1} f$, and this limit does not depend on the choice of the sequence of canonical transformations $\mathcal{A}_n$ converging to $\mathcal{A}$.

In this section we shall describe the set of bounded operators that sustain all (linear) canonical transformations in the Fermi case.

Consider the operators

$$\hat{p}(f) = \hat{a}(f) + \hat{a}^*(f), \qquad \hat{q}(f) = \frac{1}{i}(\hat{a}(f) - \hat{a}^*(f)). \tag{3}$$

For what follows it is convenient, instead of operator functionals, to consider operator generalized functions. In this connection consider a realization of the space L by means of functions with summable square on a certain set M endowed with a measure; suppose that, under this realization, the involution in L becomes complex conjugation.

Define the operator generalized function $\widehat{p}(x)$ by the equality

$$\widehat{p}(f) = \int \widehat{p}(x) f(x)\, dx.$$

The operator generalized functions $\widehat{q}(x)$, $\widehat{a}(x)$, $\widehat{a}^*(x)$ are defined analogously.

Theorem 2. *In order that a bounded operator $\hat{A}$ sustain all canonical transformations, it is necessary and sufficient that it be representable in the form of a strongly convergent series*

$$\hat{A} = \sum \frac{1}{\sqrt{m!n!}} \int K_{mn}(x_1 \dots x_m \mid y_1 \dots y_n) \times \widehat{p}(x_1) \dots \widehat{p}(x_m) \widehat{q}(y_1) \dots \widehat{q}(y_n)\, d^m x\, d^n y, \tag{4}$$

where $K_{mn}(x_1 \dots x_m \mid y_1 \dots y_n)$ are square-summable functions, antisymmetric separately in $x_1 \dots x_m$ and in $y_1 \dots y_n$, and satisfying the condition

$$\sum \int |K_{mn}(x_1 \dots x_m \mid y_1 \dots y_n)|^2\, d^m x\, d^n y < \infty.$$

The set of operators of the form (4) forms a ring, which we shall denote by $\mathfrak{B}$. In $\mathfrak{B}$ one can introduce a trace according to the formula

$$\mathrm{sp}_1 \hat{A} = K_{00}.$$

It is not difficult to verify that, with this, the usual requirement is satisfied: for any $\hat{A}_1, \hat{A}_2 \in \mathfrak{B}$,

$$\mathrm{sp}_1 \hat{A}_1 \hat{A}_2 = \mathrm{sp}_1 \hat{A}_2 \hat{A}_1.$$

Let us note that if the space L has dimension $N < \infty$, then $\mathfrak{B}$ is the ring of all matrices of order 2^N, and the trace introduced differs from the ordinary one by a factor:

$$\mathrm{sp}_1 \hat{A} = \frac{1}{2^N}\, \mathrm{sp}\, \hat{A}.$$

We give the idea of the proof of sufficiency. Introduce in $\mathfrak{B}$ the scalar product

$$(A_1, A_2) = \mathrm{sp}_1(A_1 A_2).$$

The completion of $\mathfrak{B}$ with respect to this scalar product will be denoted by $\overline{\mathfrak{B}}$.

It is not difficult to verify that every canonical transformation $\mathcal{A}$ generates a unitary operator in the Hilbert space $\overline{\mathfrak{B}}$. We denote this operator by U_A. It is not difficult to verify, further, that if $\mathcal{A}_n \to \mathcal{A}$ in the sense of the definition given in Section 1, then $U_{A_n} \to U_A$ in the strong sense.

The topology of the Hilbert space $\overline{\mathfrak{B}}$ does not coincide with any natural topology in the space of operators. Consequently, in $\overline{\mathfrak{B}}$ there may exist elements that do not correspond to operators in $\mathcal{H}$.

Let us formulate, in this connection, a general lemma which completes the proof of sufficiency and is, moreover, of independent interest.

Lemma. *Let $\hat{A}_n \in \mathfrak{B}$ be a sequence of operators whose norms are bounded by a common constant c. Then, if $\hat{A}_n \to \hat{A} \in \overline{\mathfrak{B}}$ in the sense that $(\hat{A}-\hat{A}_n, \hat{A}-\hat{A}_n) \to 0$, then $\hat{A} \in \mathfrak{B}$, $\hat{A}_n \to \hat{A}$ in the strong sense, and $\|\hat{A}\| \le c$.*

3. Let us dwell on the connection between the normal form of an operator and its representation in the form (4). Consider the exterior algebras $\mathfrak{G}_a$ and $\mathfrak{G}_p$ with generators (functions with anticommuting values) $a(x), a^*(x)$ and $p(x), q(x)$, respectively:

$$\{a(x), a^*(x')\} = \{a(x), a(x')\} = \{a^*(x), a^*(x')\} = \{p(x), p(x')\} = \{p(x), q(x')\} = \{q(x), q(x')\} = 0.$$

To the representation of each operator in normal form and in the form (4) we assign elements of the algebras $\mathfrak{G}_a$ and $\mathfrak{G}_p$ (functionals):

$$A(a^*, a) = \\ = \sum \int L_{mn}(x_1 \dots x_m \mid y_1 \dots y_n)\, a^*(x_1) \cdots a^*(x_m) a(y_1) \cdots a(y_n)\, d^m x\, d^n y; \tag{5}$$

$$\mathfrak{A}(p, q) = \sum \frac{1}{\sqrt{m!n!}} \int K_{mn}(x_1 \dots x_m \mid y_1 \dots y_n) \times \\ \times p(x_1) \cdots p(x_m) q(y_1) \cdots q(y_n)\, d^m x\, d^n y \tag{6}$$

(the functions K_{mn} in (6) are the same as in (4). For the functionals $A(a^*, a)$, see ([2])).

Using the continual integral over anticommuting variables ([2]), the connection between $A(a^*, a)$ and $\mathfrak{A}(p, q)$ can be written in the form*:

$$\mathfrak{A}(p, q) = \int \exp\left[-i \int \left(q(x) + \frac{a^*(x) - a(x)}{i\sqrt{2}}\right)\left(p(x) - \frac{a(x) + a^*(x)}{\sqrt{2}}\right) dx\right] \times \\ \times A\left(\frac{a^*}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right) \prod da^*\, da. \tag{7}$$

We omit the inversion of this formula.

The canonical transformation (1) gives rise to a relation between the operators

$$\hat{p} = \hat{a} + \hat{a}^*, \quad \hat{q} = \frac{1}{i}(\hat{a} - \hat{a}^*)$$

and

$$\hat{p}' = \hat{b} + \hat{b}^*, \quad \hat{q}' = \frac{1}{i}(\hat{b} - \hat{b}^*).$$

Replacing in this relation $\hat{p}, \hat{q}, \hat{p}', \hat{q}'$ by p, q, p', q', we obtain a linear transformation in the algebra $\mathfrak{G}_p$:

$$p' = Ap + Bq, \qquad q' = Cp + Dq,$$

where A, B, C, D are operators expressible in a definite way through Φ, Ψ.

It turns out that the functional $'\mathfrak{A}(p,q)$, corresponding to the transformed operator $'\hat{A}$, is expressed through the functional $\mathfrak{A}(p,q)$, corresponding to the operator $\hat{A}$, by the formula

$$'\mathfrak{A}(p,q) = \mathfrak{A}(p', q'),$$

i.e. the transformation of the functionals $\mathfrak{A}(p,q)$ reduces to a change of variables**.

We note that not every operator can be written in the form (4). For exam—

* Recall the definition of the integral. In the case when the Grassmann algebra $\mathfrak{G}$ has a finite number of generators $x_1, \ldots, x_N$, the integral is defined as follows: $\int dx_i = 0$, $\int x_i\, dx_i = 1$; the symbols dx_i anticommute with one another and with x_k; a multiple integral is understood as repeated. The continual integral is the limit of n-fold integrals as $n \to \infty$.

** In contrast to the functionals $\mathfrak{A}(p,q)$, the functionals $A(a^*, a)$ transform according to rather complicated formulas involving continual integration (see ([2])).

for example, the operator $\hat{A} = \int K(x,y)\hat{a}^*(x)\hat{a}(y)\, dx\, dy$ is representable in the form (4) if and only if $K(x,y)$ is the kernel of an operator with an absolutely convergent trace. The same applies to the operator

$$\hat{A} = \exp\left\{ i \int K(x,y) a^*(x) a(y)\, dx\, dy \right\}.$$

Thus, if $K(x,y)$ is the kernel of a self-adjoint, but non-nuclear operator, then $\hat{A}$ is an example of a bounded operator that does not withstand all canonical transformations.

4. Let us pass to the Bose case. Consider the linear space $\widetilde{L}$, consisting of pairs of real functions $(p(x), q(x))$. Suppose that a probability measure μ is concentrated in the space $\widetilde{L}^*$. To each operator $\hat{A}$, written in normal form, we assign a functional on $\widetilde{L}$:

$$\mathfrak{A}(p,q) = \int \exp\left\{ \frac{1}{2} \int \left[\left(p - i\frac{b+b^*}{\sqrt{2}} \right)^2 + \left(q + \frac{b^*-b}{\sqrt{2}} \right)^2 \right] dx \right\} \times \\ \times A\left(\frac{ib^*}{\sqrt{2}}, \frac{ib}{\sqrt{2}} \right) \prod db^*\, db, \tag{8}$$

where $A(a^*, a)$ is the functional corresponding to the normal form of $\hat{A}^{**}$.
We omit the inversion of formula (8).

The continual integral in this formula is understood as the limit of finite-dimensional ones, where in the finite-dimensional approximation

$$b = \xi + i\eta, \quad b^* = \xi - i\eta, \quad \prod db^* db = \pi^{-n} d^n \xi \, d^n \eta, \quad \prod dp \, dq = (2\pi)^{-n} d^n p \, d^n q.$$

Just as in the Fermi case, the canonical transformation (1) gives rise to a relation between the operators $\hat{p} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^*)$, $\hat{q} = \frac{1}{i\sqrt{2}}(\hat{a} - \hat{a}^*)$ and $\hat{p}' = \frac{1}{\sqrt{2}}(\hat{b} + \hat{b}^*)$, $\hat{q}' = \frac{1}{i\sqrt{2}}(\hat{b} - \hat{b}^*)$. Replacing in this relation $\hat{p}, \hat{q}, \hat{p}', \hat{q}'$ by p, q, p', q', we obtain a linear transformation in $\widetilde{L}$:

$$p'(x) = \int (A(x,y)p(y) + B(x,y)q(y)) \, dy + f_1(x);$$

$$q'(x) = \int (C(x,y)p(y) + D(x,y)q(y)) \, dy + f_2(x). \quad (9)$$

Theorem 3. Let the operator $\hat{A}$ correspond to a functional $\mathfrak{A}(p,q)$ that is summable with its square with respect to the measure $\mu(p,q)$, and is quasi-invariant with respect to the transformation (9). Then the operator $\hat{A}$ withstands the canonical transformation (1), and the transformed operator corresponds to the functional $'\mathfrak{A}(p,q) = \mathfrak{A}(p', q')$.

Thus, the transformation of the functional $\mathfrak{A}(p,q)$ reduces to a change of variables.

Received
10 I 1963

References

1. K. O. Friedrichs, *Mathematical Aspects of the Quantum Theory of Fields*, N. Y., 1953.
2. F. A. Berezin, DAN, 137, No. 2 (1961).
3. R. A. Minlos, Tr. Mosk. matem. obshch., 8, 497 (1959).
4. E. Wigner, Phys. Rev., 40, 749 (1932).

* Such a situation is possible if, for example, $\widetilde{L}$ is the space conjugate to a nuclear one ([3]).

** The functional $A(a^*, a)$ is defined in the same way as the analogous functional in the Fermi case, with the only difference that $a(x), a^*(x)$ are ordinary complex-valued functions ([2]). In the case of a finite number of degrees of freedom, the writing of operators by means of the functions $\mathfrak{A}(p, q)$ was first considered by Wigner ([4]).

Note: Figure translations are in progress. See original paper for figures.