DIRICHLET SERIES WITH A SEQUENCE OF COMPLEX EXPONENTS HAVING ANGULAR DENSITY

For Dirichlet series

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Abstract

Full Text

MATHEMATICS

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DIRICHLET SERIES WITH A SEQUENCE OF COMPLEX EXPONENTS HAVING ANGULAR DENSITY

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For Dirichlet series

$$f(z)=\sum_{n=1}^{\infty}a_n e^{-\lambda_n z} \tag{1}$$

with positive exponents, theorems are known on the distribution of singularities on the boundary of the half-plane of holomorphy and on the relations between the abscissa of convergence and the abscissa of holomorphy ([1]).

A. F. Leont' ev extended these theorems to sequences of Dirichlet polynomials and to Dirichlet series with complex exponents under the assumption that $\lim_{n\to\infty}\arg\lambda_n=0$ ([2]). He also obtained certain theorems for a more general type of sequences $\{\lambda_n\}$.

In the present article we consider Dirichlet series with complex exponents in the case where the sequence $\{\lambda_n\}$ has angular density ([3]), i.e., where there exists a nondecreasing function $\sigma(\varphi)$ having the property that for all φ_1,φ_2 not belonging to a certain exceptional countable set, the density

$$\lim_{k\to\infty}\frac{k}{|\lambda_{n_k}|}$$

of the sequence $\{\lambda_{n_k}\}\subset\{\lambda_n\}$, for which $\varphi_1<\arg\lambda_{n_k}<\varphi_2$, is equal to $\sigma(\varphi_2)-\sigma(\varphi_1)$. For all φ not belonging to the indicated exceptional set, the function $\sigma(\varphi)$ is continuous.

1°. The theorems on the distribution of singularities of Dirichlet series with complex exponents are based on a method of defining the domain of convergence of these series ([4]) and on the following generalization of the well-known Cramér–Pólya theorem ([1]):

If $|\arg\lambda_n|<\alpha<\pi/2$, the series (1) has a nonempty domain of convergence, the function $f(z)$ is analytic in the domain G, $\varphi(z)$ is a function of exponential type in the angle $-\alpha_1 < \arg z < \alpha_2$ $(0 < \alpha_1, \alpha_2 < \pi/2)$, and I is a domain inside which lies the conjugate diagram of the function $\varphi(z)$, then the sum of the series

$$\sum_{n=1}^{\infty} a_n \varphi(\lambda_n) e^{-\lambda_n z} = F(z) \tag{1}$$

can be analytically continued from the domain of convergence of this series into a domain G^* such that if $z_1 \in G,\ z_2 \in I$, then $z_1 - z_2 \in G^*$.

2°. Let the sequence $\{\lambda_n\}$, having angular density, be such that the set of limit points of the sequence $\{\arg \lambda_n\}$ is located on the interval $[-\alpha, \alpha]$, where $\alpha < \pi/2$, and let the condensation index of the sequence $\{\lambda_n\}$ be

$$\delta = \lim_{n\to\infty} \frac{1}{|\lambda_n|} \ln \left| \frac{1}{L'(\lambda_n)} \right|, \tag{2}$$

where

$$L(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right),$$

is finite.

Lemma. The function

$$g(z) = \sum_{n=1}^{\infty} \frac{e^{-\lambda_n z}}{L'(\lambda_n)}$$

is holomorphic outside the angles

$$\frac{\pi}{2} - \alpha \le \arg(z - z') \le \frac{\pi}{2} + \alpha, \qquad -\left(\frac{\pi}{2} + \alpha\right) \le \arg(z + z') \le -\left(\frac{\pi}{2} - \alpha\right),$$

where

$$z' = \pi(S + iC), \qquad S = \int_{\alpha-0}^{\alpha+0} \sin\varphi \, d\sigma(\varphi), \qquad C = \int_{-\alpha-0}^{\alpha+0} \cos\varphi \, d\sigma(\varphi).$$

This lemma, in combination with the above-mentioned theorem of Cramér-Pólya type, makes it possible, by constructing a corresponding function $\varphi(z)$ taking at the points λ_n the values $a_n L'(\lambda_n)$, to prove the main theorems.

In the case where the sequence $\{\lambda_n\}$ satisfies the indicated condition, we shall call an angle of holomorphy of the function $f(z)$, defined by the series (1), any angle $|\arg(z-z_0)| < \pi/2-\alpha$ inside which $f(z)$ is holomorphic, but such that no angle $|\arg(z-z_1)| < \pi/2-\alpha$, where $\operatorname{Im} z_1 = \operatorname{Im} z_0$, $\operatorname{Re} z_1 < \operatorname{Re} z_0$, has this property.

Theorem 1. If z_0 is the vertex of an angle of holomorphy of the function $f(z)$, then on the broken line with vertex z_0, consisting of the segment of the ray $\arg(z-z_0) = \pi/2-\alpha$ of length $\pi(C/\cos\alpha + S/\sin\alpha)$ and the segment of the ray $\arg(z-z_0) = -(\pi/2-\alpha)$ of length $\pi(C/\cos\alpha - S/\sin\alpha)$, there is at least one singular point of the function $f(z)$.

In the case where the density of the sequence $\{\lambda_n\}$ is equal to zero ($S=0$, $C=0$), Theorem 1 asserts that the vertex of every angle of holomorphy is a singular point of the function $f(z)$.

Corollary. If on the "upper" side of the angle of holomorphy of the function $f(z)$ there is a singular point whose distance from the vertex z_0 of this angle is greater than $\pi(C/\cos\alpha+S/\sin\alpha)$, then on every segment of length $\pi(C/\cos\alpha+S/\sin\alpha)$ between this point and the point z_0 there is at least one singular point of the function $f(z)$. The same assertion is also true for the "lower" side of the angle of holomorphy with respect to segments of length $\pi(C/\cos\alpha - S/\sin\alpha)$.

Theorem 2. Every ray lying inside an angle of holomorphy H of the function $f(z)$, making with the positive direction of the real axis an angle $\pi/2-\alpha$ and separated from the corresponding side of the angle H by a distance greater than $\pi(C\sin\alpha - S\cos\alpha)+\delta$, where δ is determined by means of (2), lies entirely, except possibly for a segment of finite length, inside the domain of convergence of the series (1).

An analogous assertion holds for a ray parallel to the other side of the angle H and separated from this side by a distance greater than $\pi(C\sin\alpha + S\cos\alpha)+\delta$.

3°. Let $\Lambda = \{\lambda_n\}$ be any sequence having angular density ($0 \le \arg\lambda_n < 2\pi$); let $\Lambda_{\varphi_1,\varphi_2}$ be the subsequence of the sequence Λ consisting of those of its terms for which $\varphi_1 \le \arg\lambda_n < \varphi_2$; let R_{φ_1,φ_2} be the series composed of those terms of the series (1) for which $\lambda_n \in \Lambda_{\varphi_1,\varphi_2}$, and let $f_{\varphi_1,\varphi_2}(z)$ be the sum of the series R_{φ_1,φ_2}. That one of the angles of opening $\pi-2\eta_0$ ($\eta_0>0$) with bisector directed along the ray $\arg z = -\varphi$, which belongs to the angles of holomorphy of all the functions $f_{\varphi,\varphi+\eta}(z)$ for $0<\eta\le\eta_0$ and which is boun-

attains the largest domain, denote it by $H_\varphi^{\eta_0}$. Let $z = l_\varphi^{(\eta_0)}e^{-i\varphi}$ be its vertex. Analogously, for the functions $f_{\varphi-\eta,\varphi}(z)$ ($0<\eta\leqslant\eta_0$) introduce the angle $H_\varphi^{-\eta_0}$ with vertex at the point $z = l_\varphi^{(-\eta_0)}e^{-i\varphi}$. Denote

$$l_\varphi^+ = \varlimsup_{\eta_0\to0} l_\varphi^{(\eta_0)}, \qquad l_\varphi^- = \varlimsup_{\eta_0\to0} l_\varphi^{(-\eta_0)}, \qquad l(\varphi) = \max(l_\varphi^+, l_\varphi^-).$$

Consider the domain K, at all points of which, for every φ $(0 \leqslant \varphi < 2\pi)$,

$$x \cos\varphi - y \sin\varphi - l(\varphi) > 0.$$

All functions of the form $f_{\varphi_1,\varphi_2}(z)$, and in particular the function $f(z)$ itself, are holomorphic in the domain K. Finally, let M be the closure of the set of singular points of all possible functions of the form $f_{\varphi_1,\varphi_2}(z)$.

Theorem 3. If the function $\sigma(\varphi)$ is continuous on the interval $[0, 2\pi]$ (in particular, if the density of the sequence $\{\lambda_n\}$ is equal to zero), then all boundary points of the domain K belong to the set M.

In the general case: a) all boundary points of the domain K that do not lie on rectilinear segments contained in this boundary belong to M; b) every rectilinear segment and every polygonal line on the boundary of the domain K of length $2\pi D$, where

$$D = \lim_{n\to\infty} \frac{n}{|\lambda_n|}$$

is the density of the sequence $\{\lambda_n\}$, contains at least one point of the set M.

Let $\delta[\varphi_1, \varphi_2)$ be the condensation index of the sequence $\Lambda_{\varphi_1,\varphi_2}$ (if this sequence is finite, then, by definition, $\delta[\varphi_1, \varphi_2) = 0$),

$$\delta^+_\varphi = \overline{\lim_{\eta\to 0}}\, \delta[\varphi, \varphi + \eta), \qquad \delta^-_\varphi = \overline{\lim_{\eta\to 0}}\, \delta[\varphi - \eta, \varphi)$$

(it can be shown that always $\delta^+_\varphi \geqslant 0$, $\delta^-_\varphi \geqslant 0$).

Theorem 4. The series (1) converges in the domain whose points, for every φ $(0 \leqslant \varphi < 2\pi)$, satisfy the condition

$$x \cos\varphi - y \sin\varphi - m(\varphi) > 0,$$

where

$$m(\varphi) = \max(l^+_\varphi + \delta^+_\varphi,\ l^-_\varphi + \delta^-_\varphi).$$

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REFERENCES

[1] V. Bernstein, *Leçons sur les progrès récents de la théorie des séries de Dirichlet*, Paris, 1933.
[2] A. F. Leont' ev, Tr. Matemat. inst. im. V. A. Steklova AN SSSR, **39** (1951).
[3] B. Ya. Levin, *Distribution of Zeros of Entire Functions*, Moscow, 1956.
[4] G. L. Lunts, Matem. sborn., **10** (52), No. 1-2 (1942).

Note: Figure translations are in progress. See original paper for figures.