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Abstract

Full Text

MATHEMATICS

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ON SYSTEMS OF DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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In the theory of equations with constant coefficients, the questions relating to a single equation with one unknown function and to a square system of equations reducible to it, with determinant different from identically zero, have by now been studied in considerable detail. In particular, the “global” theory for such equations is close to completion, i.e. the totality of problems concerning solvability, the general form, local properties, etc. of solutions of these equations outside any boundary-value problems. At the same time, very little is known about general systems of equations, including “underdetermined” and “overdetermined” systems.

In the present note we shall deal with transferring the “global” theory to such systems; namely, we shall formulate theorems on solvability and the general form of solutions of general systems. This problem, as in the case of one equation, splits into two essentially quite different problems: solvability and the general form of solutions of systems in spaces of (ordinary or generalized) functions with growth less than exponential, and in spaces in which exponential growth is allowed.

The first problem was studied in ^(4,9). A criterion for the solvability of general systems of equations in the space S' (with right-hand sides from the same space) was found in ⁽⁴⁾, and also, in a dual formulation, in ⁽⁹⁾ for arbitrary spaces of functions of power growth. In ⁽⁹⁾ the general form of solutions of systems in such spaces is also given.

For spaces in which exponential growth is allowed, solutions of these problems were first formulated by L. Ehrenpreis ⁽³⁾; however, his formulation of the theorem on the general form of solutions of systems turned out to be incorrect. Recently a number of results in this direction were obtained independently by B. Malgrange and by the author. We shall now formulate the theorems obtained by us on solvability and on the general form of solutions of general systems in spaces in which exponential growth is allowed.

Let us write a general system of linear equations with constant coefficients:

$$\begin{aligned} p_{11}(D)u_1 + \dots + p_{1m}(D)u_m &= w_1, \\ &\dots \\ p_{k1}(D)u_1 + \dots + p_{km}(D)u_m &= w_k. \end{aligned} \tag{1}$$

Here $D = (i\partial/\partial x_1, \dots, i\partial/\partial x_n)$, and the p_{ij} are polynomials. We shall denote by $p(D)$ the matrix of operators; for convenience put $u = (u_1, \dots, u_m)$, $w = (w_1, \dots, w_k)$. We shall choose the right-hand sides and seek solutions in the spaces Φ of the following collection: the spaces $S_\alpha^{\beta'}$ and \mathcal{E}_α^β , $0 \leq \alpha \leq 1$, $0 \leq \beta \leq \infty$, of generalized, respectively infinitely differentiable, functions defined in the whole space with various growth conditions at infinity (see ^(6,8)); the spaces $S_0^{\beta'}(\Omega)$ and $\mathcal{E}_0^\beta(\Omega)$, $1 < \beta \leq \infty$, of generalized, respectively infinitely differentiable, functions defined in an arbitrary open convex domain Ω (see ⁽¹⁰⁾).

All the results that we shall formulate are equally applicable to any of these spaces.

The product of q copies of the space Φ will be denoted by $[\Phi]^q$.

Let us first consider the question of the solvability of system (1). We note that, in order for system (1) to be solvable, it is necessary that every linear relation with coefficients of the form $q_1(D), \dots, q_k(D)$, where q_1, \dots, q_k are polynomials, which holds between the rows of the matrix $p(D)$, should also hold between the right-hand sides w . The converse is also true.

Theorem 1. *For the solvability in $[\Phi]^m$ of system (1) with $w \in [\Phi]^k$, it is necessary and sufficient that, for any operators $q_1(D), \dots, q_k(D)$ with constant coefficients such that $\sum q_i p_{ij} = 0$ for every j , the relation $\sum q_i w_i = 0$ be satisfied.*

It is known from algebra that, in order to verify the fulfillment of this condition, it is enough to use a finite number of suitably chosen vectors q_1, \dots, q_k .

A special case of this theorem, concerning one equation and the space \mathcal{D}' of all generalized functions of finite order, was obtained as early as 1953 by Malgrange ⁽¹⁾ and Ehrenpreis ⁽²⁾.

For the space $\mathcal{E}_0(\Omega)$ of infinitely differentiable functions in an arbitrary convex domain Ω , a somewhat different solvability criterion was obtained independently by Malgrange ⁽⁵⁾.

We now pass to the problem of describing the general form of the solutions of (1) with $w = 0$. We note that this problem reduces to its special case concerning a system of equations with one unknown function

$$p_1(D)u = \dots = p_k(D)u = 0. \tag{2}$$

Moving in (1) the terms with u_2, \dots, u_m to the right-hand side and considering the resulting equations as equations with respect to u_1 , we apply to them the solvability criterion described in Theorem 1. We obtain a finite system of equations with respect to u_2, \dots, u_m . Knowing the general form of the solutions of this system, we can find the general solution of system (1). Indeed, for every solution (u_2, \dots, u_m) of the new system we can, by Theorem 1, find a function u_1 such that the row $u = (u_1, \dots, u_m)$ satisfies (1); moreover, the general form of such functions u_1 is the sum of a particular solution u_1 and the general solution of a system of the form (2) with $p_i = p_{i1}$, $i = 1, \dots, k$. With the new system we can carry out the same operation, singling out the function u_2 , and so on.

Thus, let us consider system (2). Performing the Fourier transform, we arrive at the system

$$p_1(s)\tilde{u} = \dots = p_k(s)\tilde{u} = 0 \quad (3)$$

in the dual space $\tilde{\Phi}$. For all spaces Φ that we consider here, $\tilde{\Phi}$ is the space of entire functions φ of n variables $s = (s_1, \dots, s_n)$ satisfying the system of inequalities

$$|\varphi(s)| \leq c_{\varphi, \alpha} M_{\alpha}(z), \quad (4)$$

where $\{M_{\alpha}(z)\}$ is a certain (depending on Φ), generally speaking uncountable, family of positive functions.

Let \mathfrak{p} be the polynomial ideal generated by the polynomials p_1, \dots, p_k , and let $\mathfrak{p} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_{\rho}$ be its representation as an intersection of primary ideals. For each ideal \mathfrak{q}_{μ} we can choose a splitting of the variables s into two groups $s = (\xi_{\mu}, \eta_{\mu})$ such that, for every subspace of the form $\eta_{\mu} = \eta_{\mu}^0$, the ideal $\mathfrak{q}_{\mu}(\eta_{\mu}^0)$, formed by the restrictions to this subspace of polynomials belonging to \mathfrak{q}_{μ} , is a zero-dimensional ideal.

The well-known theorem of Max Noether asserts that, in order that a polynomial $f(\xi_{\mu})$ belong to the zero-dimensional ideal $\mathfrak{q}_{\mu}(\eta_{\mu}^0)$, it is necessary and

it is sufficient that at each root ξ_{μ}^0 of this ideal a finite number of linear relations be satisfied among the derivatives $f(\xi_{\mu})$. The meaning of these conditions is that the polynomial $f(\xi_{\mu})$ must be a linear combination of polynomials belonging to $\mathfrak{q}_{\mu}(\eta_{\mu}^0)$, with coefficients that are formal power series in $\xi_{\mu} - \xi_{\mu}^0$. These conditions can be written in the form of the vanishing, at the point ξ_{μ}^0 , of a certain finite collection of differential operators applied to $f(\xi_{\mu})$. The number and orders of these operators are bounded at all points $s^0 = (\xi_{\mu}^0, \eta_{\mu}^0)$ belonging to the variety N_{μ} of the roots of the ideal \mathfrak{q}_{μ} . We shall call these operators Noetherian and denote them by $d_{\mu}^i(s^0, D)$, $|i| \leq \sigma_{\mu}$, $\mu = 1, \dots, \rho$.

It is clear that every exponential polynomial which is the Fourier transform of a functional of the form $d_{\mu}^i(s^0, D)\delta(s - s^0)$, $s^0 \in N_{\mu}$, is a solution of (1). Indeed,

every functional of this form vanishes on the ideal \mathfrak{p} , and consequently on all functions of the form $p_1\varphi_1 + \dots + p_k\varphi_k$, $\varphi_1, \dots, \varphi_k \in \widetilde{\Phi}'$, and thus satisfies the system (3). It turns out that the converse is also true: every solution of the system (2) can be composed from such exponential polynomials.

It can be shown that if we discard from the variety N_μ a sufficiently large neighborhood of the set of its singular points, then on the remaining set N_μ^0 , under a proper normalization of the Noetherian operators, their coefficients will be rational functions with moduli growing no faster than a certain power of $|s| + 1$. Thus, for any function $\varphi \in \Phi'$, the functions $d_\mu^i(s, D)\varphi(s)$ on the set N_μ^0 satisfy estimates of the form (4).

Theorem 2. *The general form of the solutions of the system (2) is given by the equality*

$$(u, \varphi) = \sum_{i, \mu} \int_{N_\mu^0} d_\mu^i(s, D) \tilde{\varphi}(s) d\lambda_\mu^i(s), \quad \varphi \in \Phi',$$

where $d\lambda_\mu^i(s)$ are arbitrary measures concentrated on N_μ^0 and defining continuous functionals on $\widetilde{\Phi}'$, i.e. such that $\int M_\alpha(s) |d\lambda_\mu^i(s)| < \infty$ for all α .

This theorem generalizes and refines analogous results (⁷, ¹⁰) concerning a single equation. Malgrange in (⁵) independently obtained the following result: every solution of (2) in the space $\mathcal{D}'(\Omega)$ of generalized functions of finite order, concentrated in an arbitrary convex domain Ω , is the Fourier transform of a functional of finite order concentrated on the variety $N_{\mathfrak{p}} = \bigcup_{\mu} N_\mu$ of common roots of the polynomials p_1, \dots, p_k . However, as is clear from Theorem 2, not every such functional is the Fourier transform of a solution of the system (2).

Let us note that in the general case we cannot construct a system of Noetherian operators with constant coefficients, as is asserted in essence in (³). This can be verified from the following example: $n = 3$, $p_1(s) = s_1^2$, $p_2(s) = s_2^2$, $p_3(s) = s_3s_1 - s_2$, $N_{\mathfrak{p}}$ is the s_3 -axis. We may put, for example, $d^0(s, D) \equiv 1$, $d^1(s, D) = \partial/\partial s_1 + s_3\partial/\partial s_2$. Any other system of Noetherian operators consists of combinations of these operators and, consequently, contains operators with variable coefficients.

The theorems formulated allow one to carry over to arbitrary systems many results known for a single equation, in particular, to investigate questions of hypoellipticity and hyperbolicity of such systems. The role which, for one equation, is played by the variety of roots

of the corresponding polynomial, for the system (1), will be played by the union of the varieties N_φ corresponding to all systems of the form (2) obtained in the course of the transformation of the system (1) described above, or, what is the same thing, the variety N on which the rank of the matrix $p(s)$ is less than the number of unknown functions.

In this note we shall only observe that all the results of ⁽⁸⁾ concerning a single equation carry over to the general system if we replace the variety of roots of the corresponding polynomial by the variety N (moreover, in the formulations of the theorems of ⁽⁸⁾, the restriction $\alpha > 0$ may even be replaced by $\alpha \geq 0$).

In conclusion, the author expresses his gratitude to B. Malgrange for providing the opportunity to become acquainted with his work ⁽⁵⁾ before its publication.

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